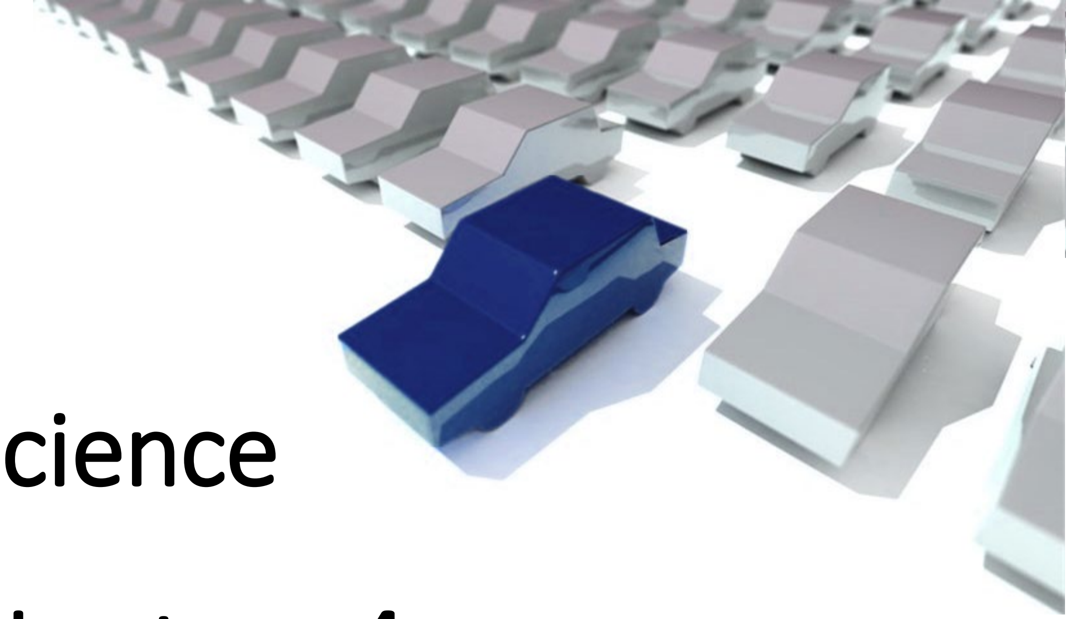

CEE 551 Traffic Science

Traffic Flow Theory Lecture 4

Godunov scheme and Cell Transmission Model

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Outline

- ❑ Godunov scheme: numerical solution of LWR model
- ❑ Cell transmission model

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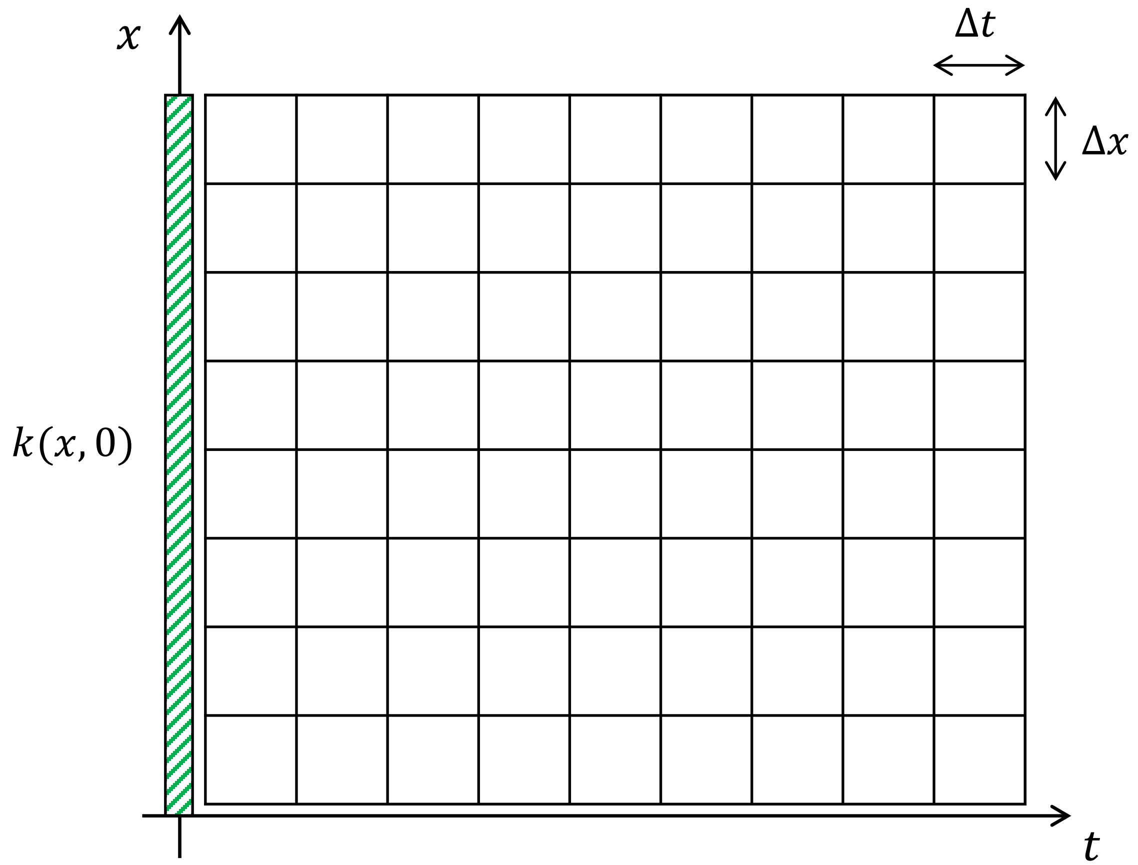
Numerical solution

- Discretize the temporal and spatial space according to given resolutions (Δx and Δt)
- Find the density of each grid given boundary conditions



- Density is defined at each road segment and each time step:

$$k(x,t) \equiv k(x\Delta x, t\Delta t)$$

- Solve the LWR model through stepping:

$$k(x,t+1) = f\big(k(x,t)\big)$$

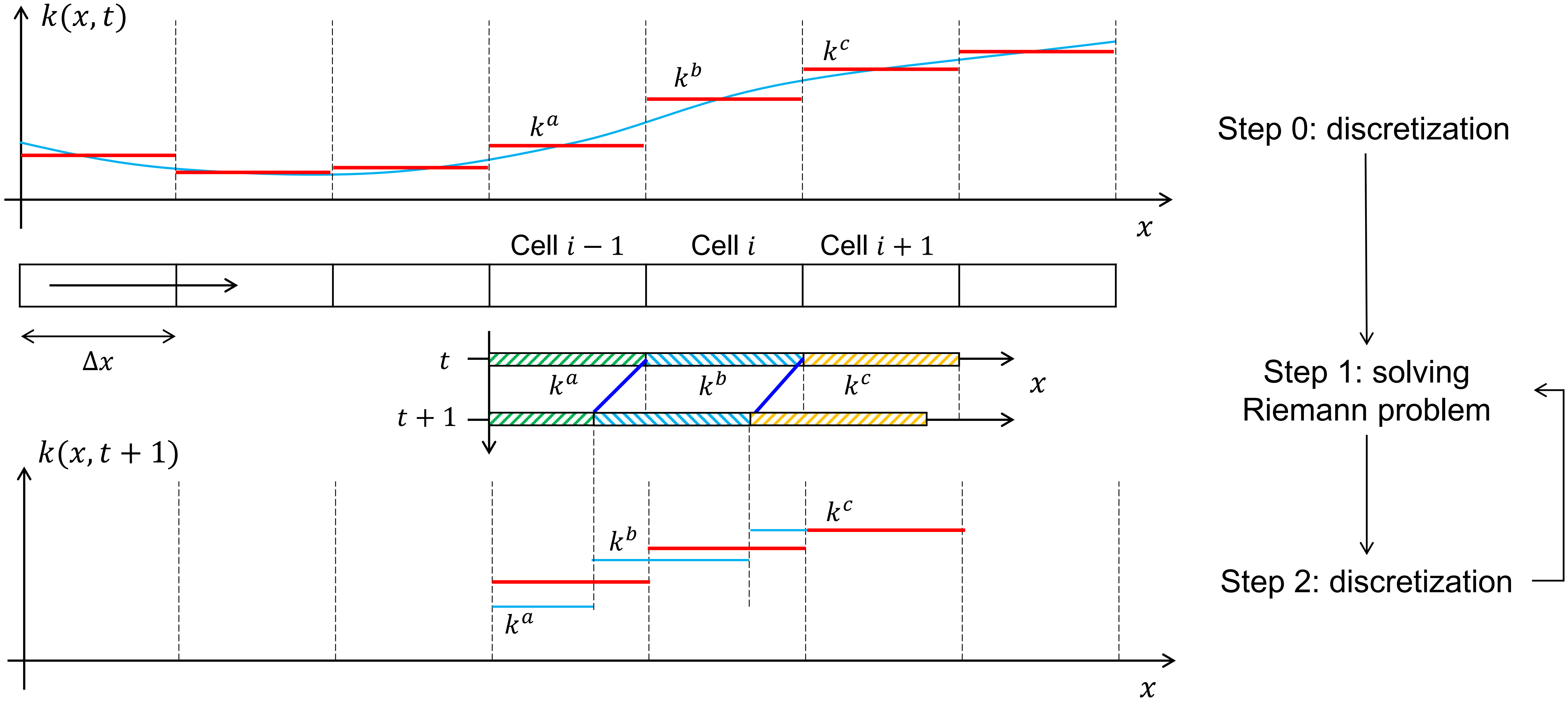
- (Dynamical equation)

Why do we need a numerical solution?

- We already showed the analytical solutions to the LWR model (e.g., Riemann problem, shockwave theory), why do we still need a numerical solution?

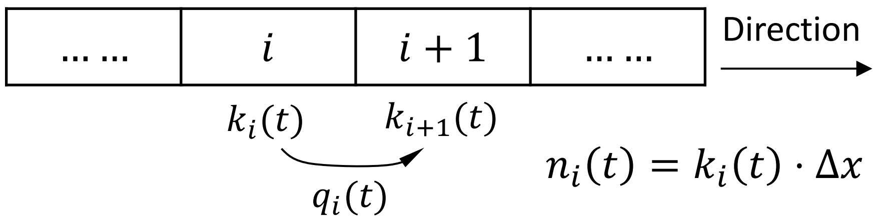
- Numerical solution has the following advantages
  - It can deal with more complicated boundary conditions (e.g., what if the initial density is a general curve instead of a step function)
  - Numerical solution essentially provides the system dynamics and can be directly used as a macroscopic traffic simulation (some researchers prefer the term "mesoscopic")
  - It is more suitable to be executed by a computer program (in mini-project 1, you will need to program yourself to implement it)

Intuition of Godunov scheme



General framework of Godunov scheme

- Step 1: find the boundary flow for every two adjacent cells



$$n_i(t) = k_i(t) \cdot \Delta x$$

  - $q_i(t)$ is a function of $k_i(t)$ and $k_{i+1}(t)$, which can be derived by solving the Riemann problem

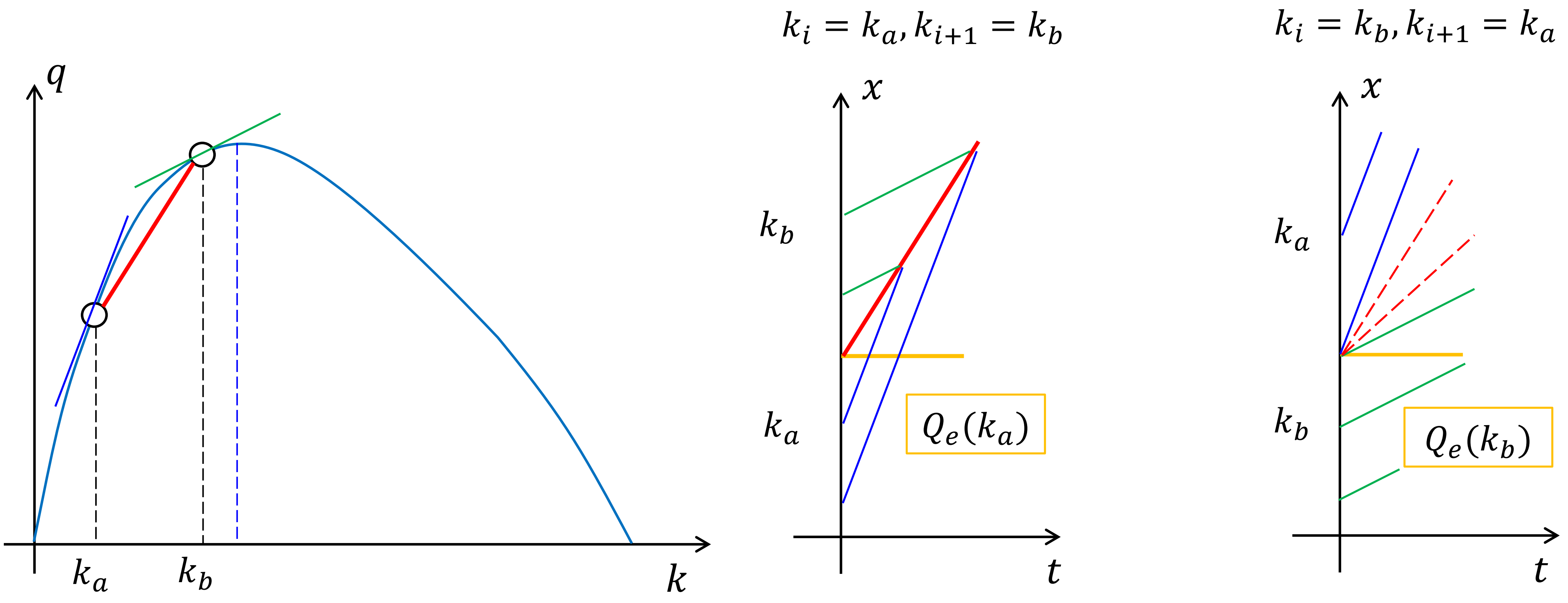
- Step 2: update the density of each cell according to the conservation law

$$n_i(t + \Delta t) = n_i(t) + q_i(t)\Delta t - q_{i+1}(t)\Delta t$$

$$k_i(t + \Delta t) = \frac{1}{\Delta x} \cdot n_i(t) + \Delta t$$

Derivation of the boundary flow

- Example case: $k_{i+1} \leq k_c$ and $k_i \leq k_c$



For both cases, the boundary flow is $Q_e(k_i)$

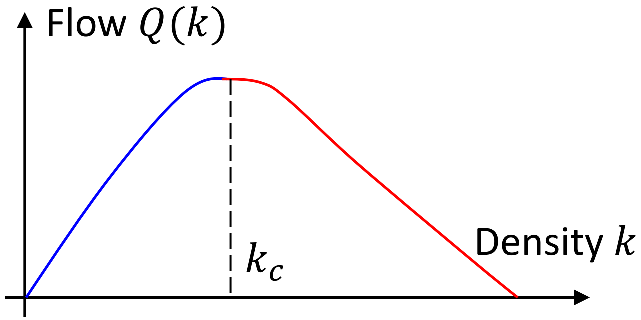
Derivation of the boundary flow (cont'd)

- Boundary flow of two adjacent cells given different cases

Downstream cell density (columns); Upstream cell density (rows)

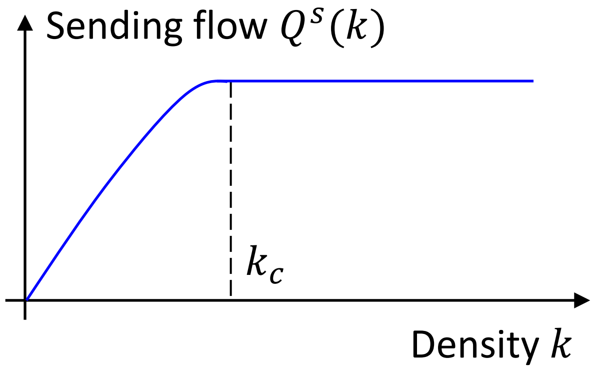
q_i	$k_{i+1} \le k_c$	$k_{i+1} > k_c$
$k_i \le k_c$	$\boldsymbol{Q_e(k_i)}$	$\min\{Q_e(k_i), Q_e(k_{i+1})\}$
$k_i(x) > k_c$	q_{max}	$Q_e(k_{i+1})$

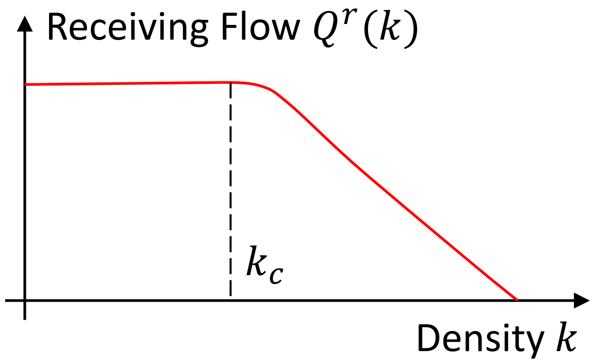
Derivation of other cases is available in the TFT_document.pdf



- It is easy to verify this is consistent with

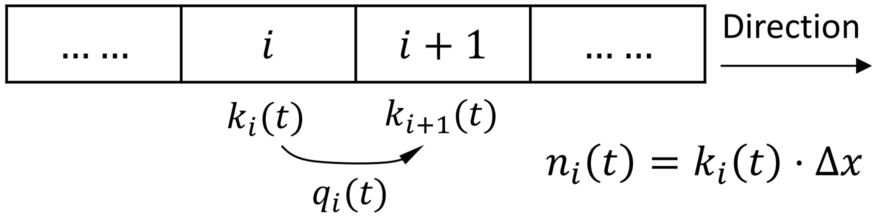
$$q_i = \min\left(Q^s(k_i), Q^r(k_{i+1})\right)$$





Godunov scheme

- Final procedure of Godunov scheme



$$n_i(t) = k_i(t) \cdot \Delta x$$

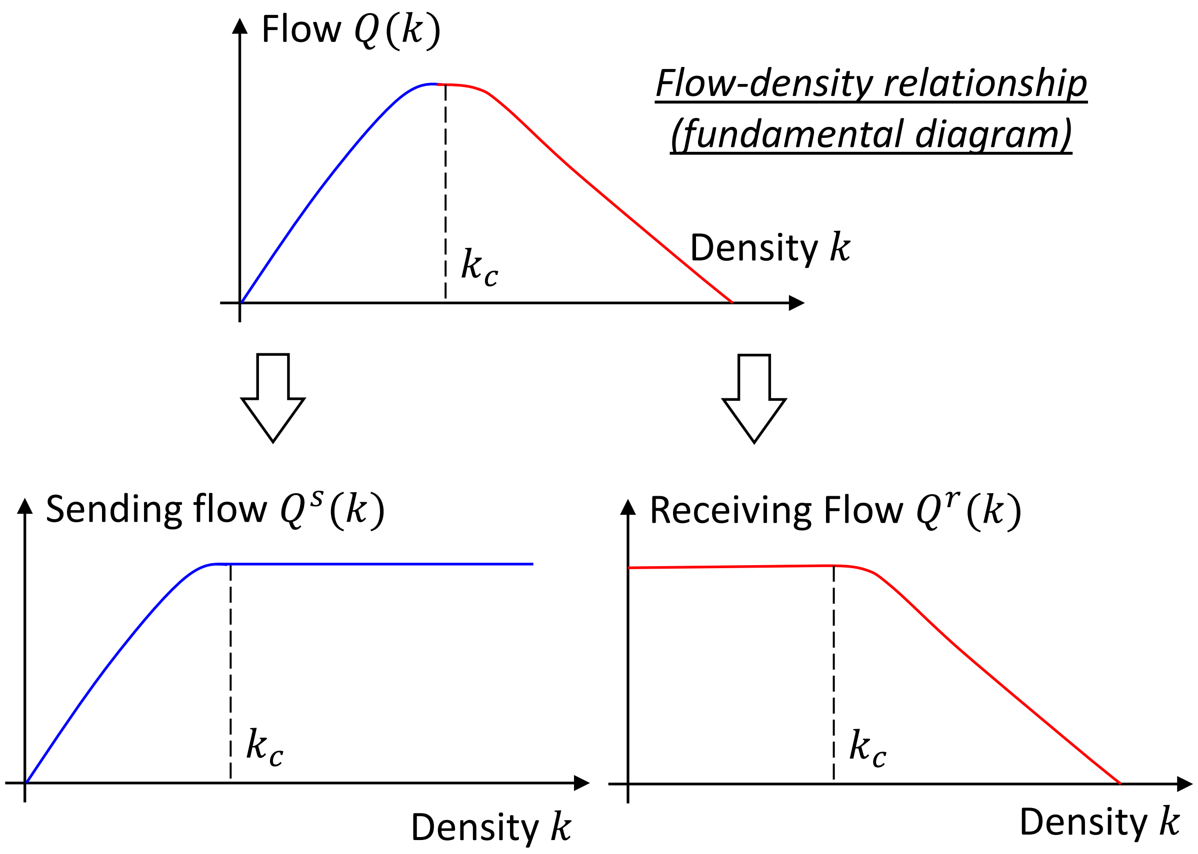
1. Get the boundary flow between cells

$$q_i(t) = \min\left(Q^s\big(k_i(t)\big), Q^r\big(k_{i+1}(t)\big)\right)$$

2. Update the density according to the conservation law

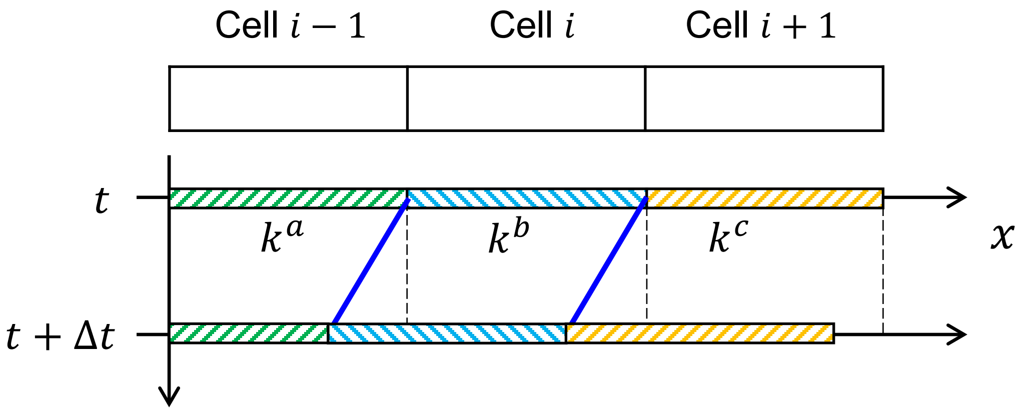
$$n_i(t+\Delta t) = n_i(t) + q_i(t)\Delta t - q_{i+1}(t)\Delta t$$

$$k_i(t+\Delta t) = \frac{1}{\Delta x} \cdot n_i(t) + \Delta t$$



A critical issue with Godunov scheme

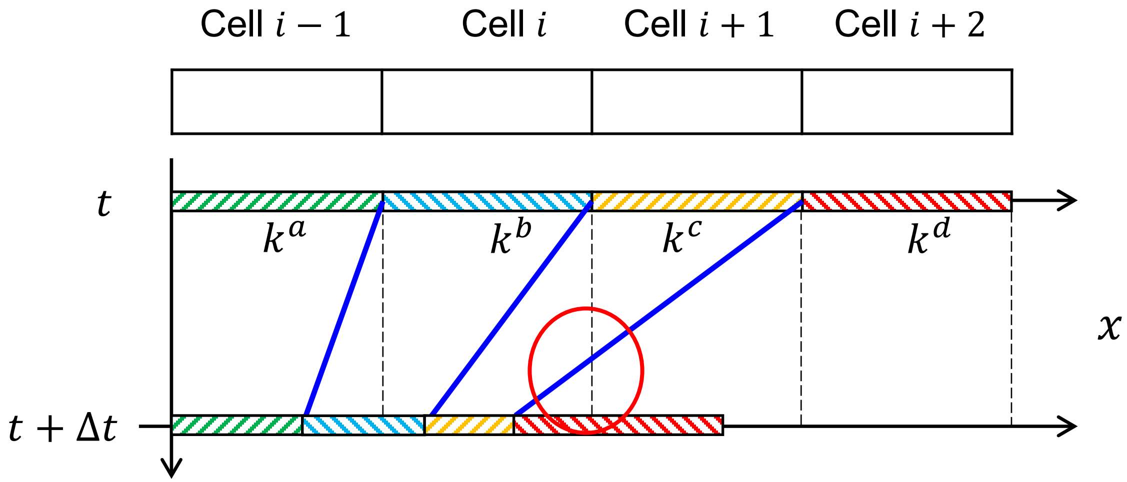
- Godunov scheme does not work when the shockwaves or rarefaction fans cross the cell boundary within one time step



- When updating the density (number of vehicles of cell i), it is *always* sufficient for us take inputs from the upstream and downstream cells

$$n_i(t+\Delta t) = f\left(n_{i-1}(t), n_i(t), n_{i+1}(t)\right)$$

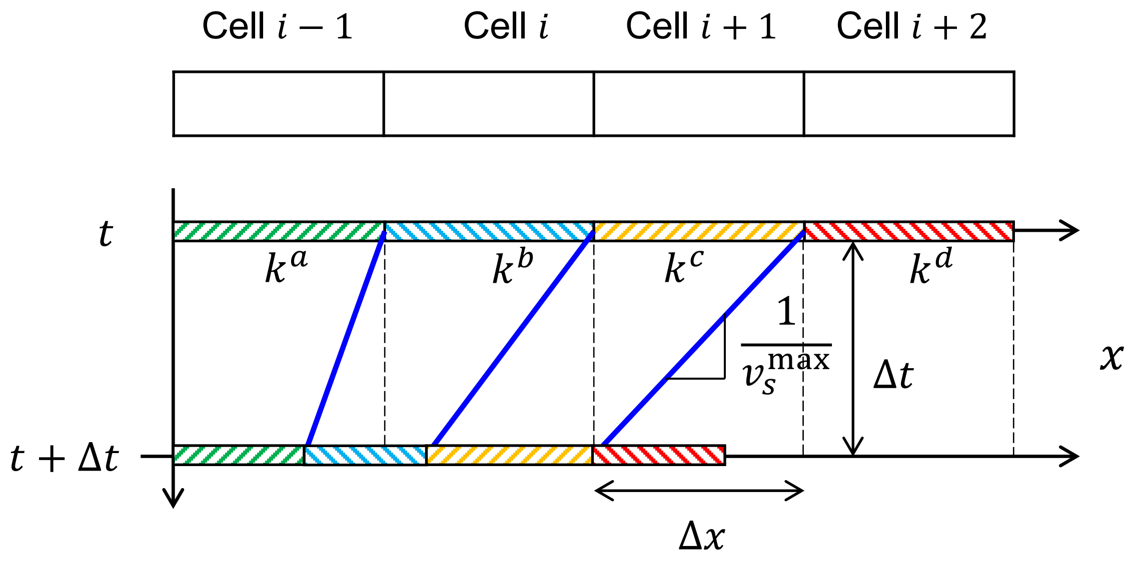
- However, if Δt is too large, shockwaves (or rarefaction fans) could cross the boundary

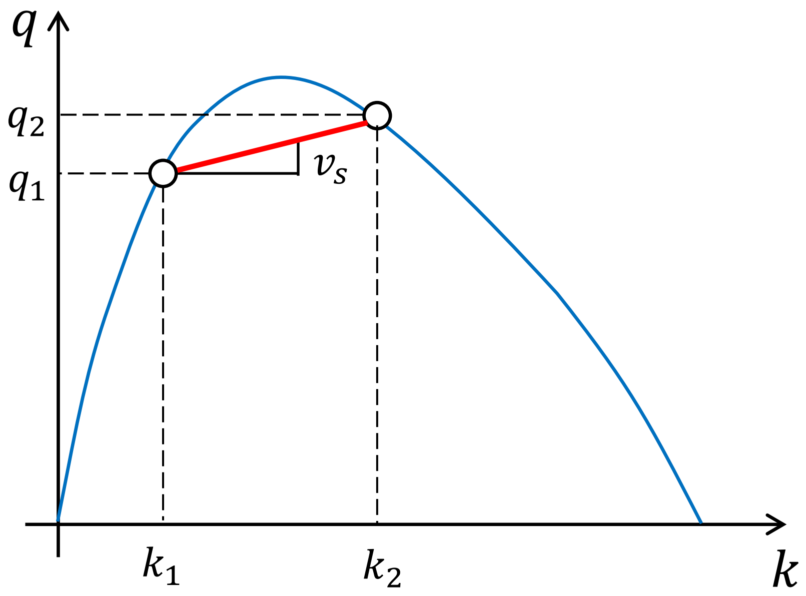


- In this case, it is not sufficient to take neighbor cells as input

CFL condition

- Courant, Friedrichs, and Lewy condition: under the maximum shockwave (or rarefaction fan) speed, it will not cross the cell boundary





$$|v_s^{max}| \le \frac{\Delta x}{\Delta t} \quad \Rightarrow \quad \max|Q_e'(k)| \le \frac{\Delta x}{\Delta t}$$

(CFL condition)

Outline

- ❑ Godunov scheme: numerical solution of LWR model
- ❑ Cell transmission model

Cell transmission model

- ❑ Cell transmission model is a specific implementation of the Godunov scheme. It is much easier to understand and implement (one of the most well-known models)
- ❑ It was proposed by Prof. Carlos Daganzo (1994, 1995, see reading materials)

- UM alumnus (PhD at Civil 1975 and MS at Civil 1973)
- Chancellor's Professor at UC Berkeley

ISTTT 22 @ Northwestern (2017.7)

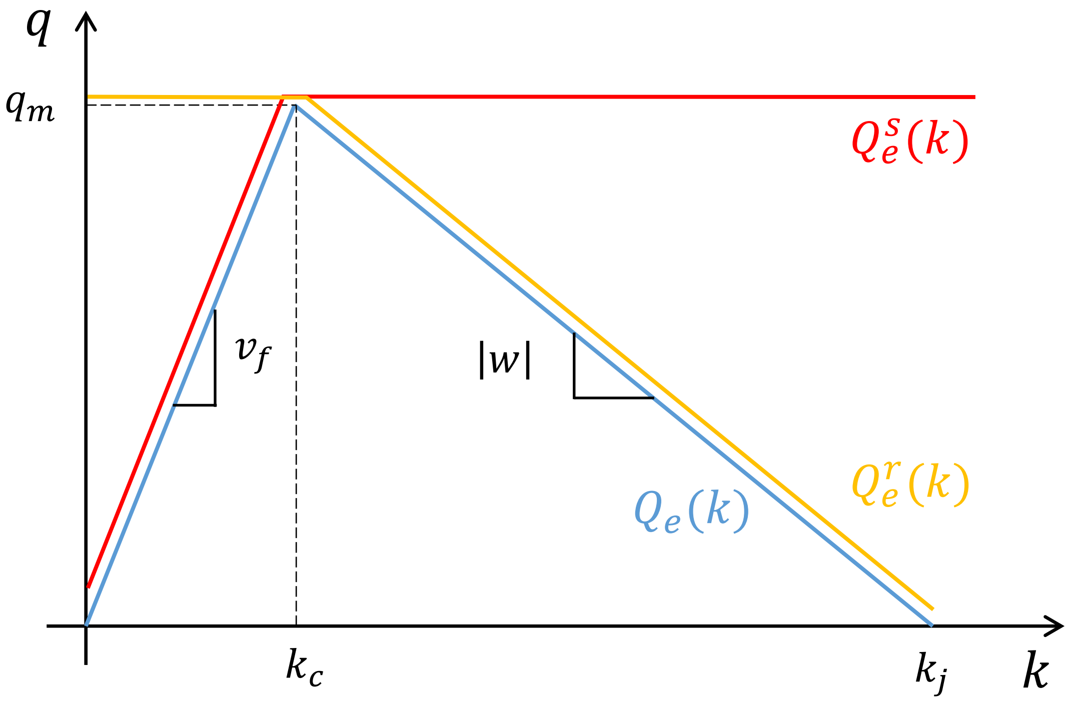
Cell transmission model

- CTM is a specific implementation of the Godunov scheme
  - Use the number of vehicles in each cell as the traffic state of each time
  - Use a triangular fundamental diagram
  - Choose Δt and Δx to satisfy the CFL condition (at bound)

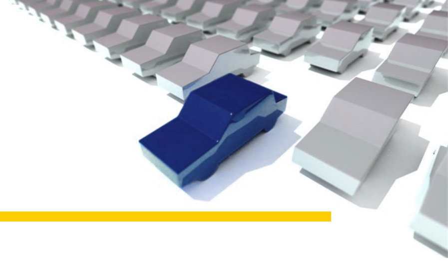
$$\max|Q_e'(k)| \leq \frac{\Delta x}{\Delta t} \Rightarrow v_f = \frac{\Delta x}{\Delta t}$$

$$\Delta x = v_f \Delta t$$

The length of each cell equals the free-flow speed times the time interval

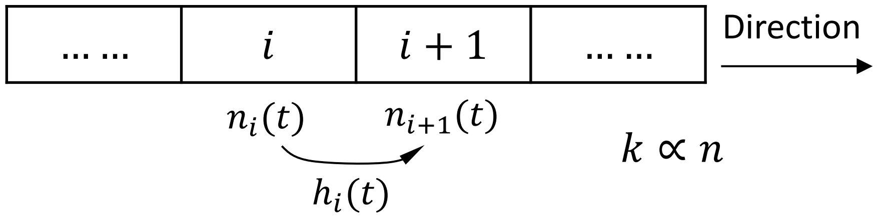


Cell transmission model



❑ CTM is a specific implementation of the Godunov scheme

- Notations of CTM
  - $n_i(t)$: number of vehicles in cell i at time step t
  - $h_i(t)$: boundary flow (# of vehicles) between cell i and cell $i+1$



- Conservation law

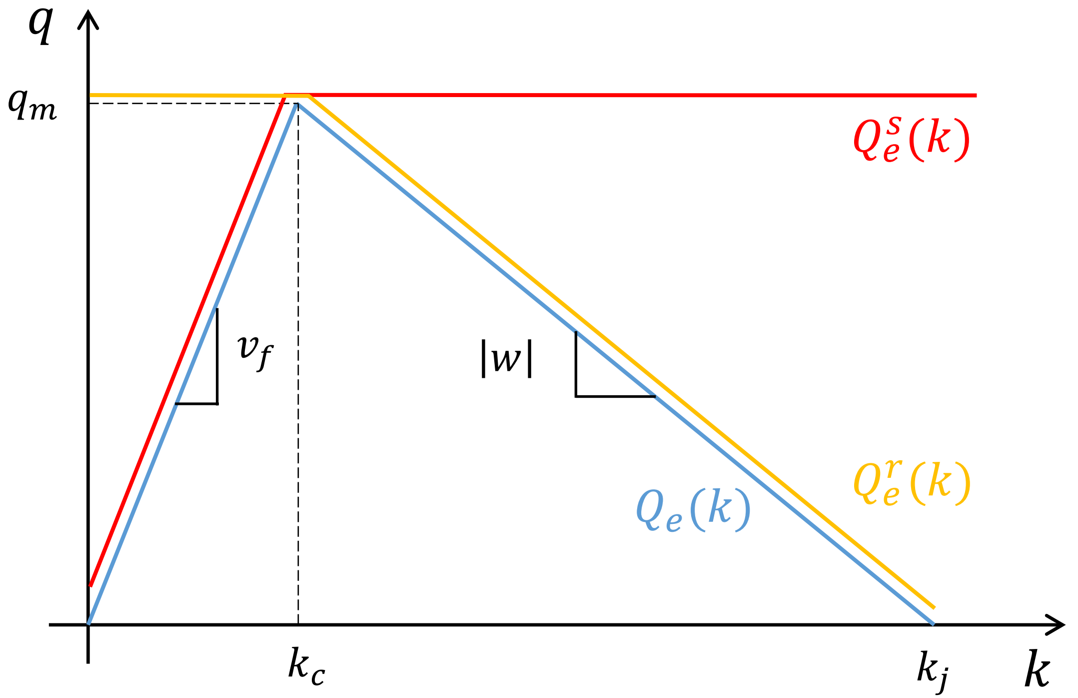
$$n_i(t+1) = n_i(t) + h_{i-1}(t) - h_i(t)$$

- Boundary flow calculation

$$h_s(n) = \min\{q_m \Delta t, v_f k \Delta t\} = \min\{q_m \Delta t, n\}$$

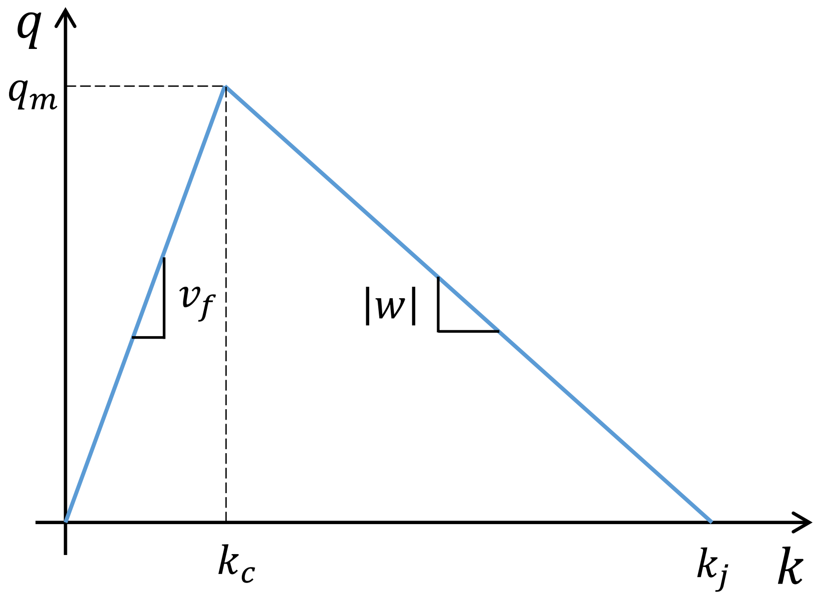
$$h_r(n) = \min\left\{q_m \Delta t, \frac{|w|}{v_f}\left(N_{jam} - n\right)\right\}$$

⇨ $$h_i(t) = \min\left\{n_i(t), Q_m, \frac{|w|}{v_f}\left(N_{jam} - n_{i+1}(t)\right)\right\}$$



FD parameters and CTM parameters

- CTM parameters can be determined given FD parameters



$$h_i(t) = \min\left\{n_i(t), Q_m, \frac{|w|}{v_f}\left(N_{jam} - n_{i+1}(t)\right)\right\}$$

v_f	Free-flow speed	m/s
w	Shockwave speed	m/s
k_c	Critical density	$veh/meter$
k_j	Jam density	$veh/meter$
q_m	Maximum flow	veh/sec

$$q_m = v_f \cdot k_c$$

$$|w| = \frac{q_m}{k_j - k_c}$$

CTM parameters

Δt	Time interval	/	sec
Δx	Cell length	$= v_f \Delta t$	m
w, v_f	/	$= w, v_f$	m/s
Q_m	Maximum flow per time step	$= q_m \Delta t$	$veh/step$
N_{jam}	Maximum # of vehicles per cell	$= k_j \Delta x$	$veh/cell$

Implementation of CTM

- Building your CTM model (initialization)
  - Choose a proper time interval Δt and length of the road segment Δx such that:

$$\Delta x = v_f \Delta t$$

  - Establish your CTM model such as number of cells, cell connections
- CTM stepping
  - Step 1: get the boundary flow (in units of # of vehicles) for each cell connection

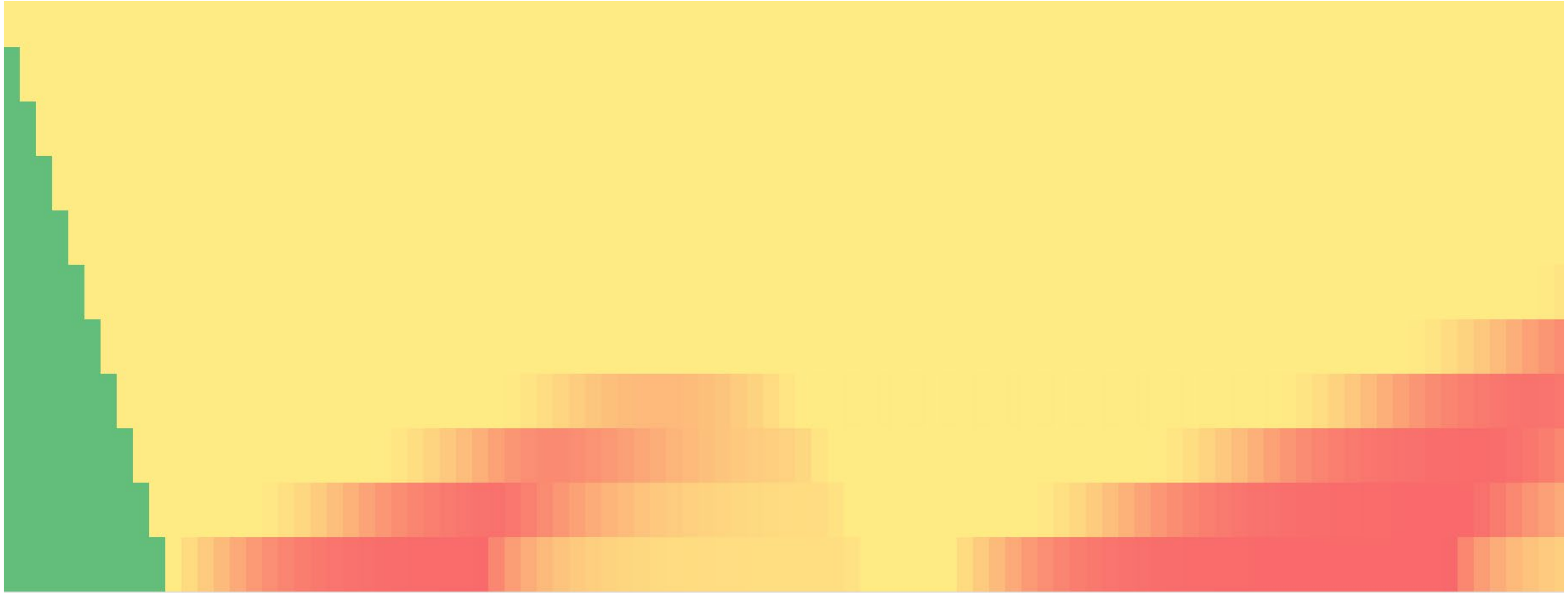
$$h_i(t) = \min\left\{n_i(t), Q_m, \frac{|w|}{v_f}\left(N_{jam} - n_{i+1}(t)\right)\right\}$$

  - Step 2: update the number of vehicles according to the conservation law

$$n_i(t+1) = n_i(t) + h_{i-1}(t) - h_i(t)$$

CTM example

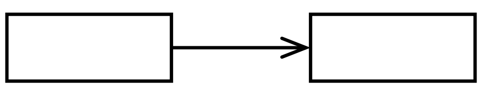
- A road with a single lane controlled by a fixed-time traffic signals



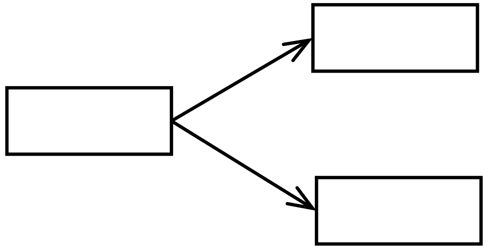
(Implemented by Excel)

CTM – general connections

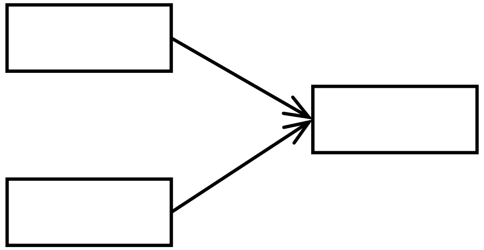
- ❑ To model a general network, there are some other connection types:



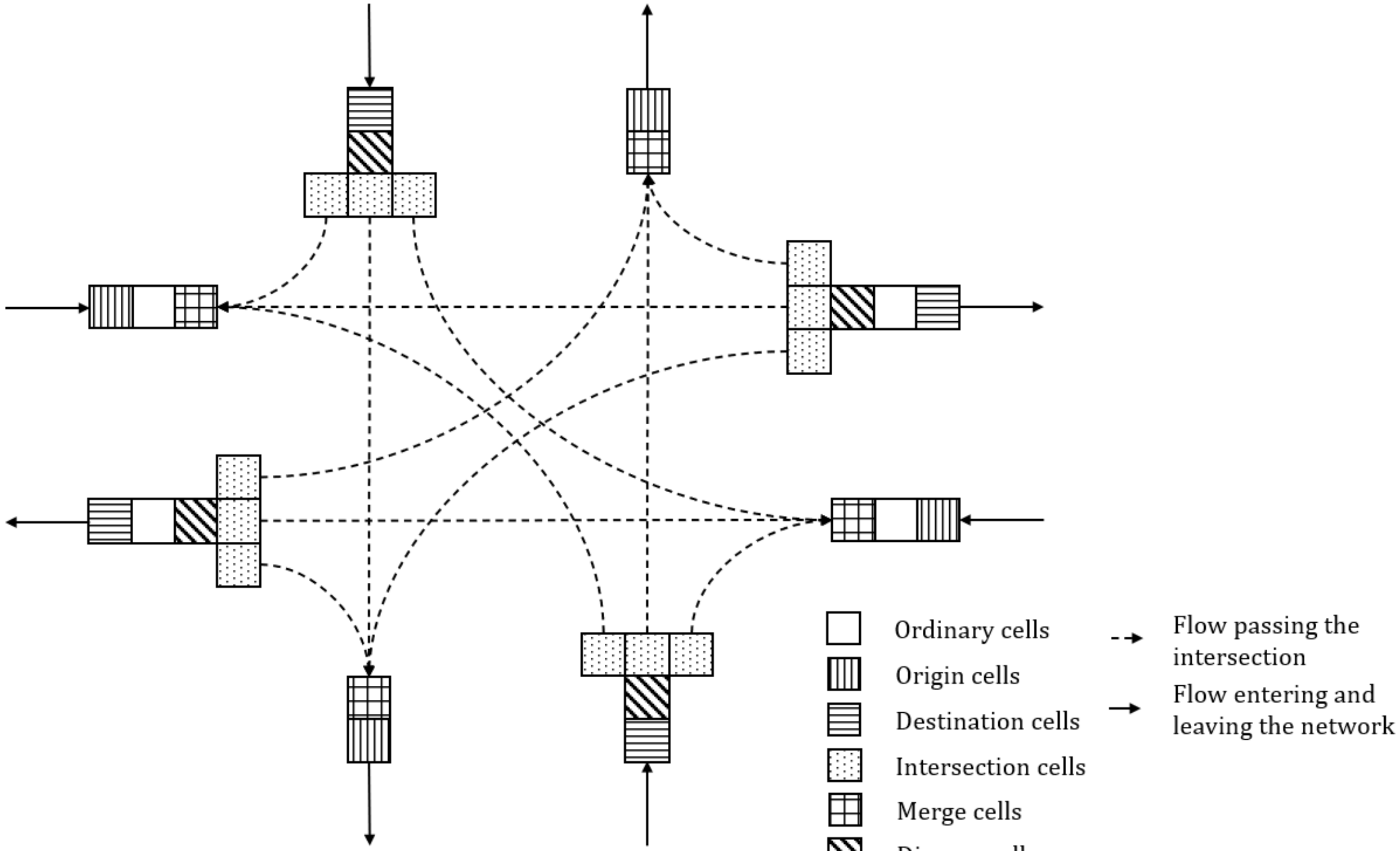
Ordinary



Diverge



Merge



Reading

- TFT_Document.pdf Section 6
- Daganzo, Carlos F. "The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory." Transportation research part B: methodological 28.4 (1994): 269-287.
- Daganzo, Carlos F. "The cell transmission model, part II: network traffic." Transportation Research Part B: Methodological 29.2 (1995): 79-93.

Mini-project 1 – traffic flow theory

- Mini-project 1 is posted today and due on Oct. 2.
- This is a group work. Each group has up to two students (working alone is not suggested). You need to submit a report as well as your implementation (code, etc.)
- We will have some of you present on Oct. 3 (only those groups need to present need prepare slides). If you volunteer to present the mini-project 1, email me (xingminw@umich.edu) with the presenter's name and the group member before Sept. 18. I only accept volunteers until the time slots are filled. If not fulfilled in the end, I will assign randomly

- We will have three mini-projects throughout this semester, and each student needs to present at least once
- By principle, one student represents the whole group to give a complete presentation (we do not suggest two students split the presentation)

Tour to Macomb County Traffic Operation Center

- Thursday, September 26th (no class on that day), 3:00 – 4:30 PM
- It is not mandatory, email zjerome@umich.edu (Zachary Jerome) if you cannot make it
- Carpool amongst classmates recommended (email zjerome@umich.edu if you have trouble getting there)

- Link to Macomb County Department of Roads Traffic Operations Center