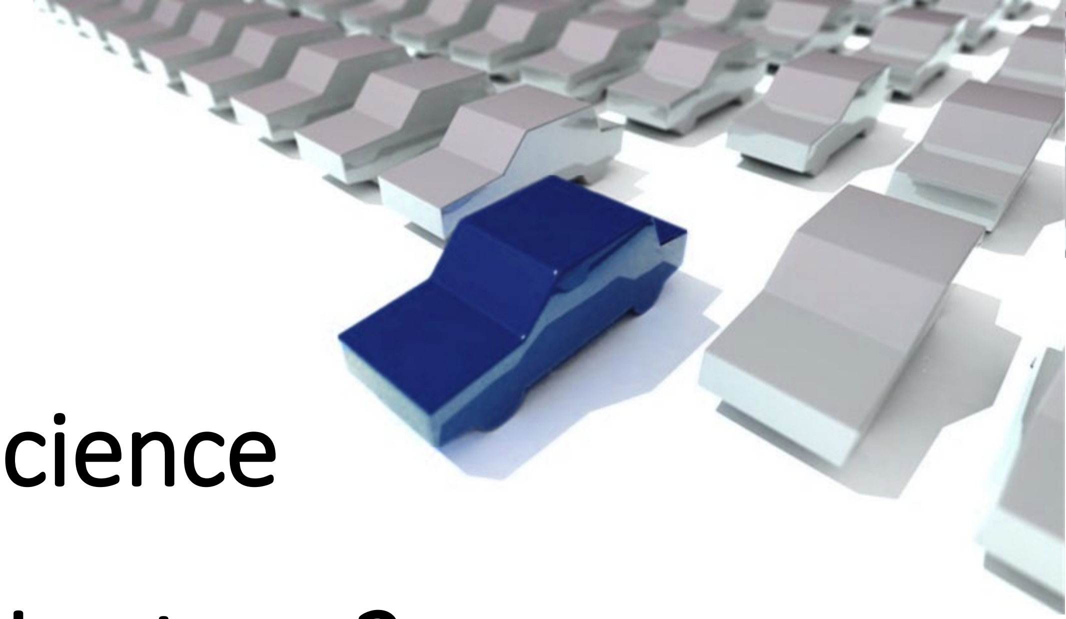

CEE 551 Traffic Science

Traffic Flow Theory Lecture 2

LWR model and its solution

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Macroscopic traffic flow variables

Name	Notation	Meaning	Units
Average density	$\bar{k}(x,t)$	The average number of vehicles per unit length of the road at position x and time t	$\frac{[veh]}{[distance]}$
Average flow rate	$\bar{q}(x,t)$	The average number of vehicles to cross position x per unit time at time t	$\frac{[veh]}{[time]}$
Average speed	$\bar{v}(x,t)$	The mean speed of vehicles at position x at time t	$\frac{[distance]}{[time]}$

$$\bar{q}(x,t) = \bar{k}(x,t) \cdot \bar{v}(x,t)$$

Microscopic traffic flow variables

Name	Notation	Meaning	Units
(Time) headway	$h_j(x) = t_j(x) - t_{j-1}(x)$	Time interval between two consecutive crossing times at position x	$\frac{[time]}{[veh]}$
Spacing	$s_j(t) = x_{j-1}(t) - x_j(t)$	Distance between leading and following vehicles at time t	$\frac{[distance]}{[veh]}$

- Relationship between macro- & micro- traffic flow variables

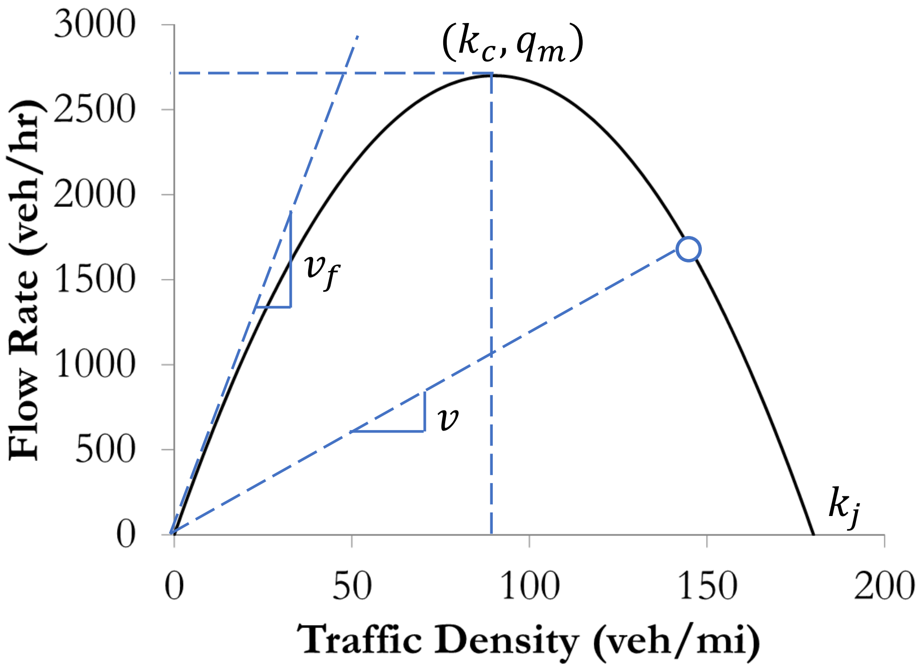
$$\bar{k}(x,t) \cdot \bar{s}(x,t) = 1$$

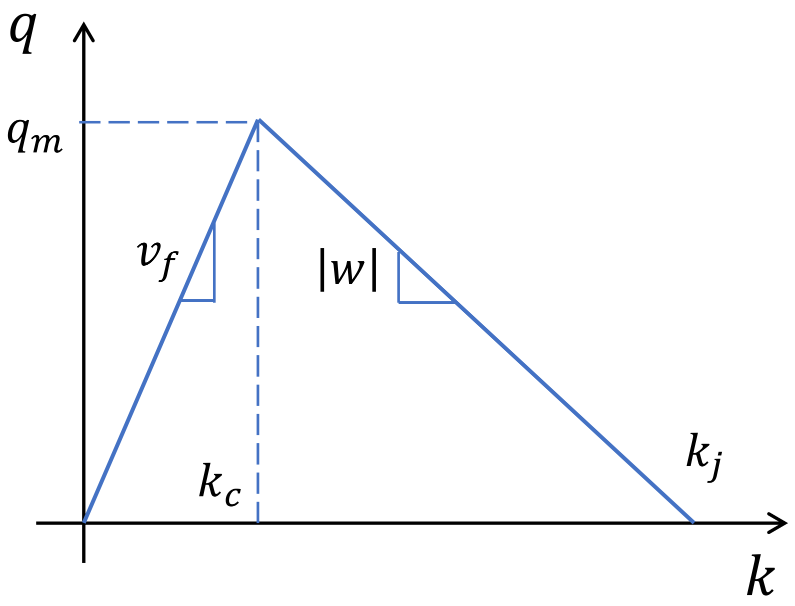
$$\bar{h}(x,t) \cdot \bar{q}(x,t) = 1$$

Example: a headway of 2 sec/veh corresponds to a flow rate of 1800 veh per hour

Fundamental diagram

- Greenshields' fundamental diagram & triangular fundamental diagram

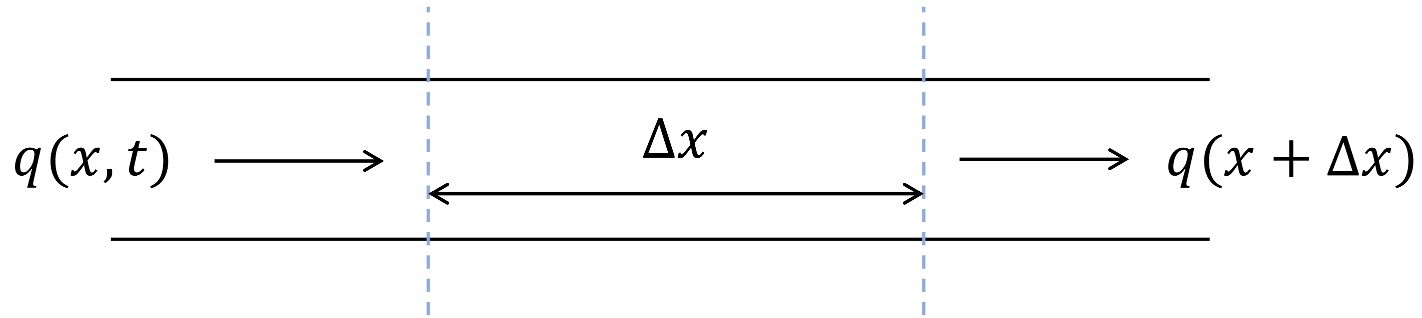




v_f	Free-flow speed
k_c	Critical density
q_m	Maximum flow
k_j	Jam density
w	Shockwave speed

Conservation law

- Let $[x, x+\Delta x]$ be an arbitrary road segment



$$k(x,t+\Delta t)\Delta x = k(x,t)\Delta x + q(x,t)\Delta t - q(x+\Delta x)\Delta t$$

$$\Rightarrow \quad \frac{k(x,t+\Delta t) - k(x,t)}{\Delta t} + \frac{q(x+\Delta x) - q(x,t)}{\Delta x} = 0$$

$$\Rightarrow \quad \frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0$$

Traffic flow relationship

- Traffic flow physics

$$q(x,t) = k(x,t) \cdot v(x,t)$$

- Conservation law

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0$$

- Empirical observation (fundamental diagram)

$$q(x,t) = Q_e\big(k(x,t)\big)$$

Outline

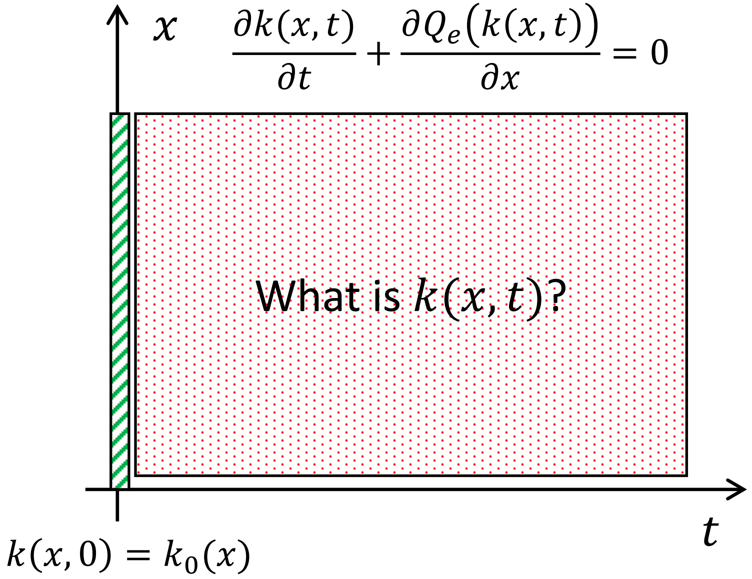
- ❑ LWR model
- ❑ Characteristic line
- ❑ Shockwave and its speed
- ❑ Riemann problem, shockwave, and rarefaction fan

LWR model

- LWR model formulation
  - Combining the conservation law and flow-density empirical relationship, we will have the LWR (Lighthill-Whitham-Richards) model:

$$\left.\begin{aligned} \frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \\ q(x,t) = Q_e\big(k(x,t)\big) \end{aligned}\right\} \quad \frac{\partial k(x,t)}{\partial t} + \frac{\partial Q_e\big(k(x,t)\big)}{\partial x} = 0$$

  - Initial state (boundary condition): $k(x,0) = k_0(x)$



- Solving the LWR model
  - Solving the PDE is to get $k(x,t)$ at any time t and location x given the location x

Characteristic lines

- Derivation of characteristic lines

$$\frac{\partial Q_e(k(x,t))}{\partial x} = \frac{\partial Q_e(k(x,t))}{\partial k(x,t)} \cdot \frac{\partial k(x,t)}{\partial x}$$

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial Q_e(k(x,t))}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial k(x,t)}{\partial t} + Q_e'(k(x,t)) \cdot \frac{\partial k(x,t)}{\partial x} = 0$$

$$dk(x,t) = \frac{\partial k(x,t)}{\partial x} dx + \frac{\partial k(x,t)}{\partial t} dt$$ (total derivative)

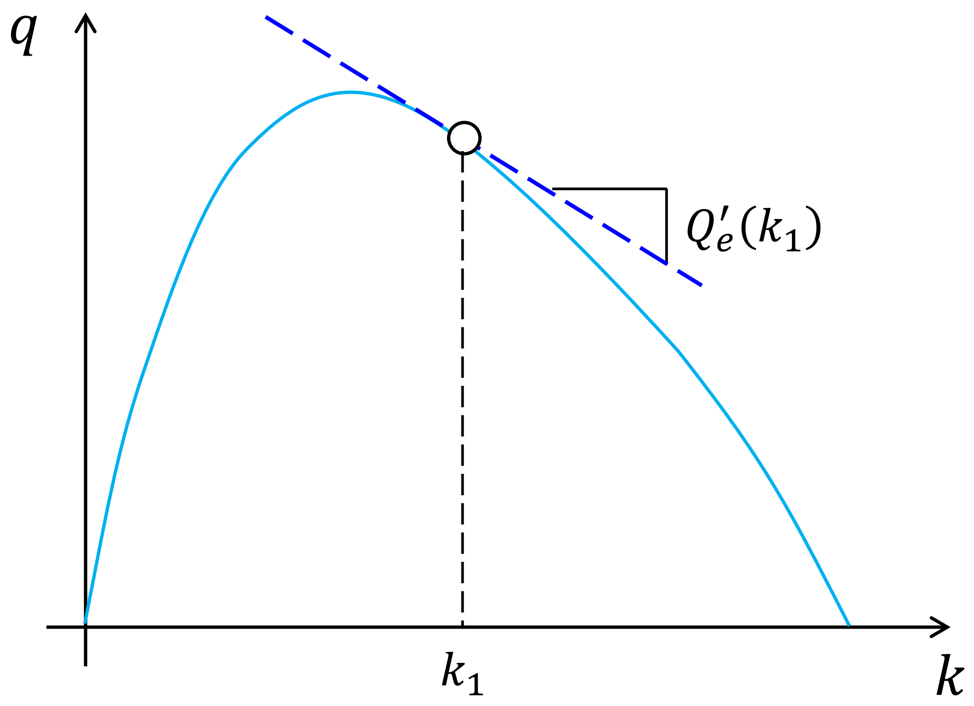
$$dk(x,t) = \frac{\partial k(x,t)}{\partial x} dx - Q_e'(k(x,t)) \cdot \frac{\partial k(x,t)}{\partial x} dt = \frac{\partial k(x,t)}{\partial x}\left[1 - Q_e'(k(x,t))\frac{dt}{dx}\right] dx$$

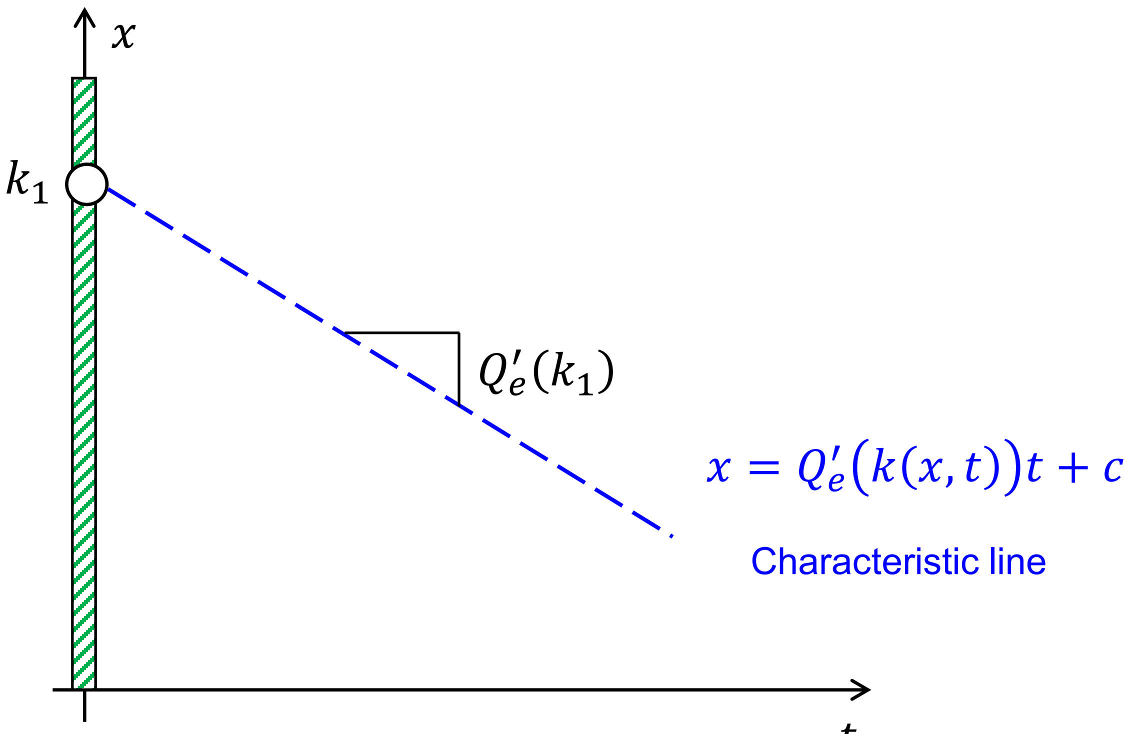
$$1 - Q_e'(k(x,t))\frac{dt}{dx} = 0 \rightarrow \frac{dx}{dt} = Q_e'(k(x,t)) \quad \Rightarrow \quad dk(x,t) = 0$$

$x = Q_e'(k(x,t))t + c$ Traffic density is the same along this line (characteristic line)

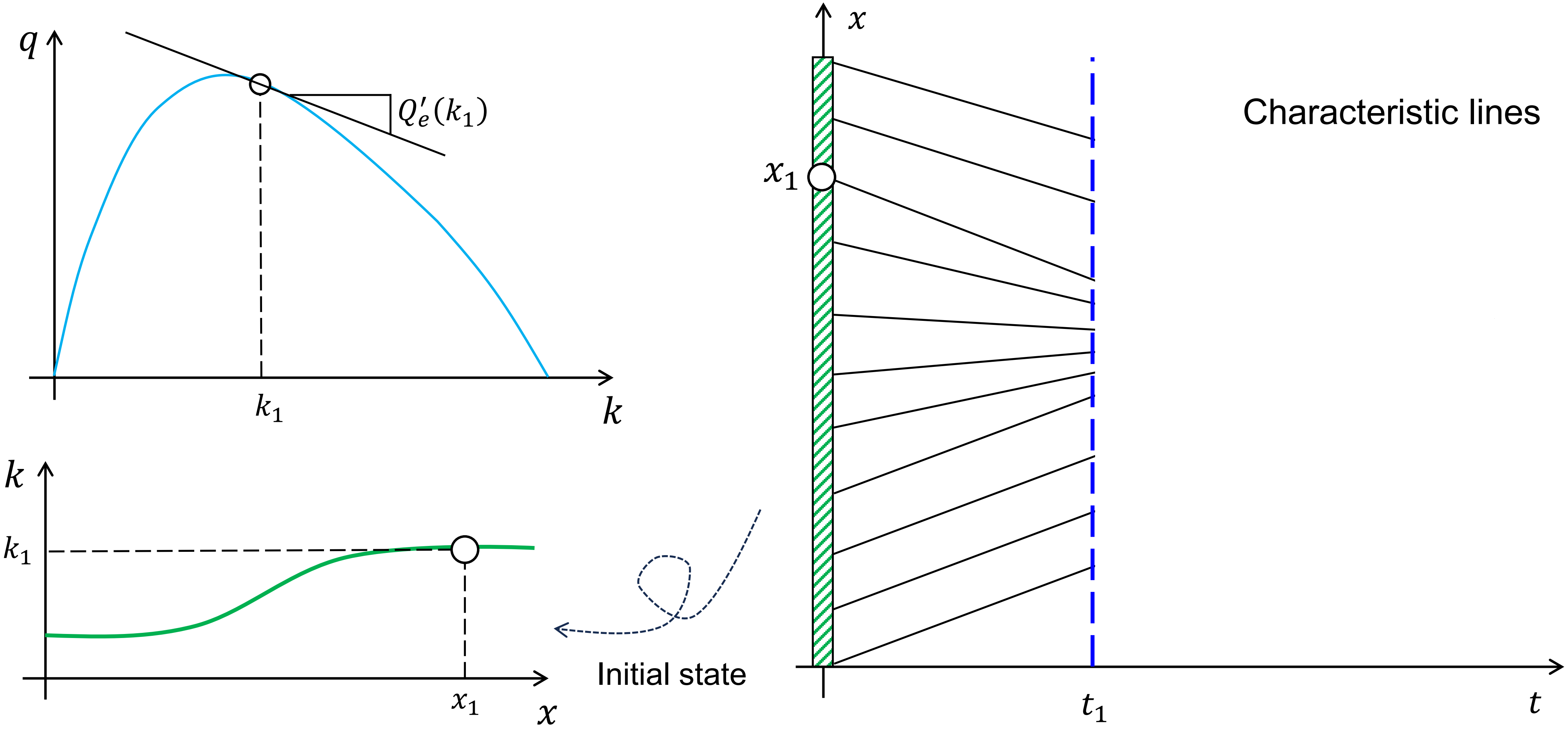
Illustration of characteristic lines

- Characteristic line speed is a different concept with vehicle speed

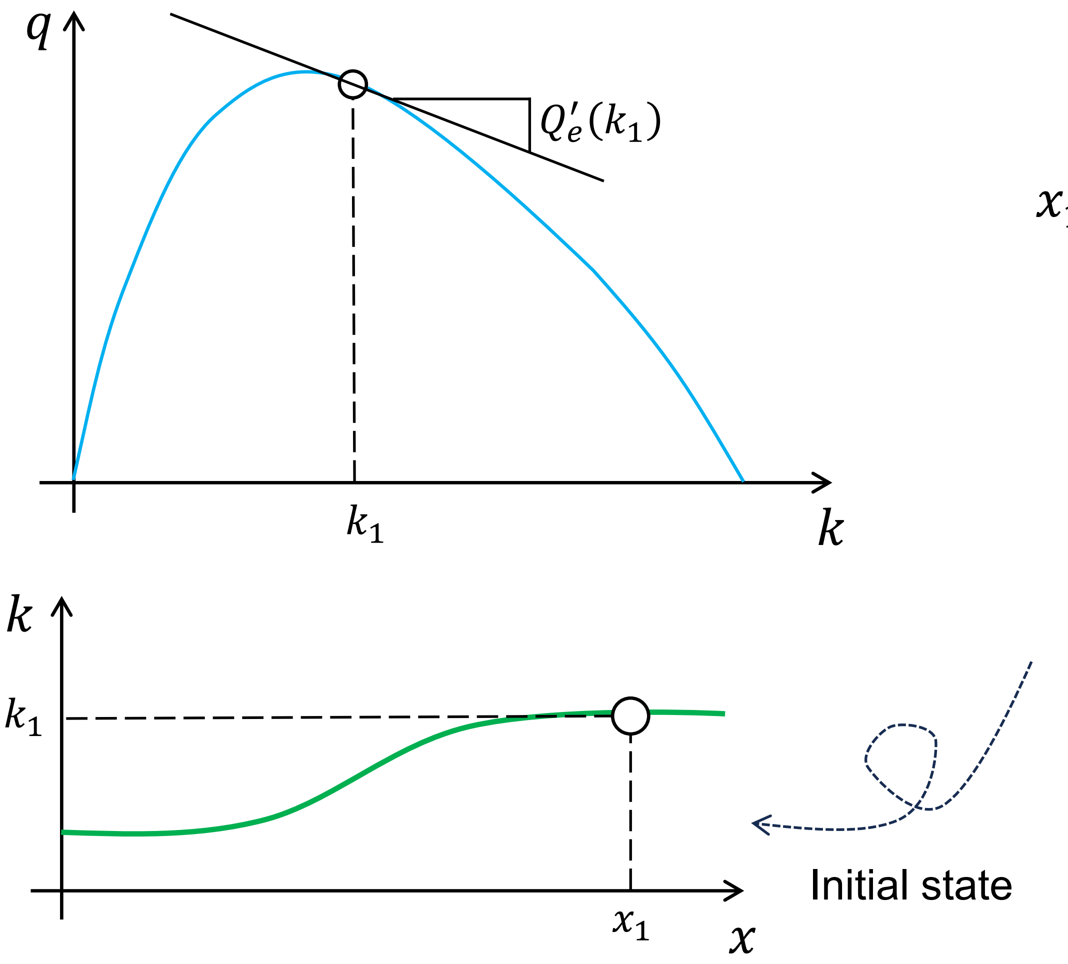


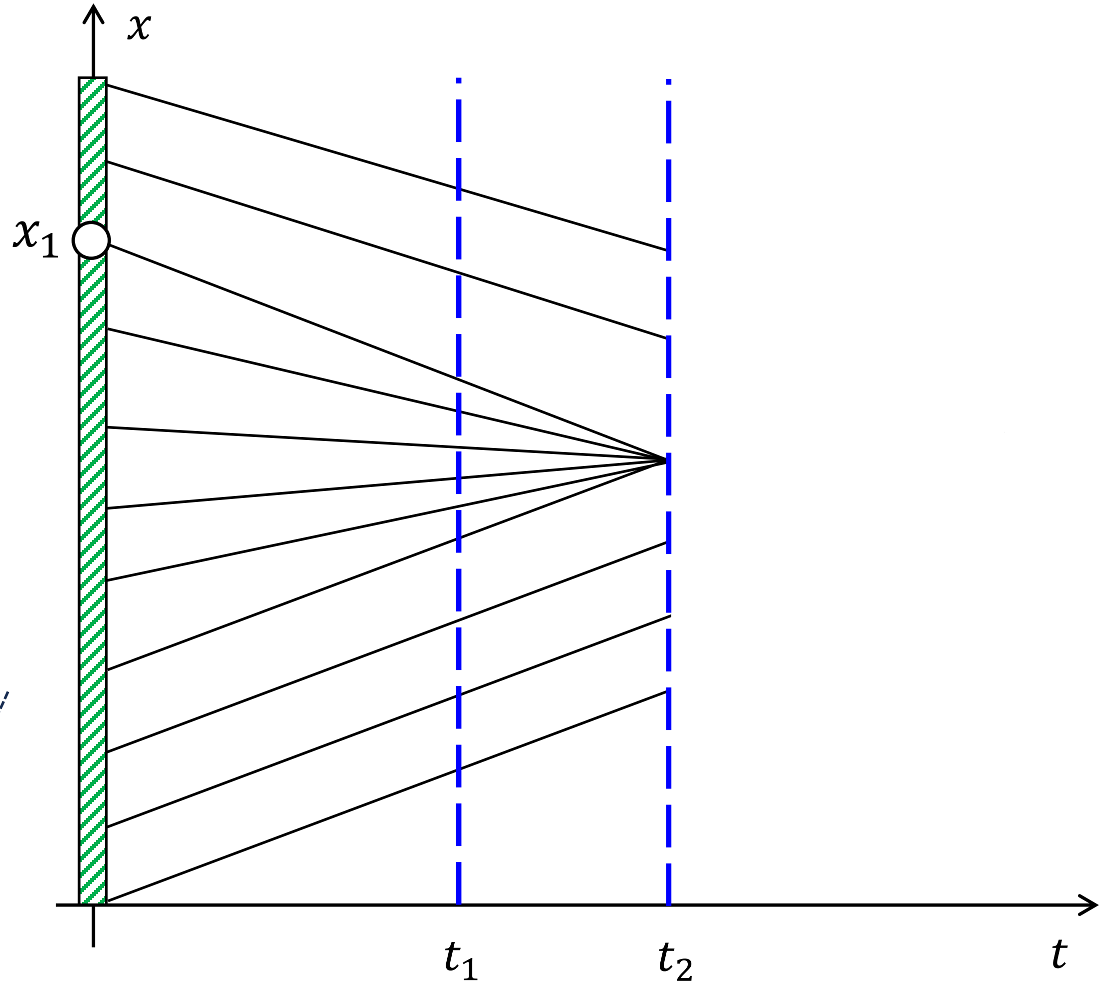


Solving LWR model with characteristic line

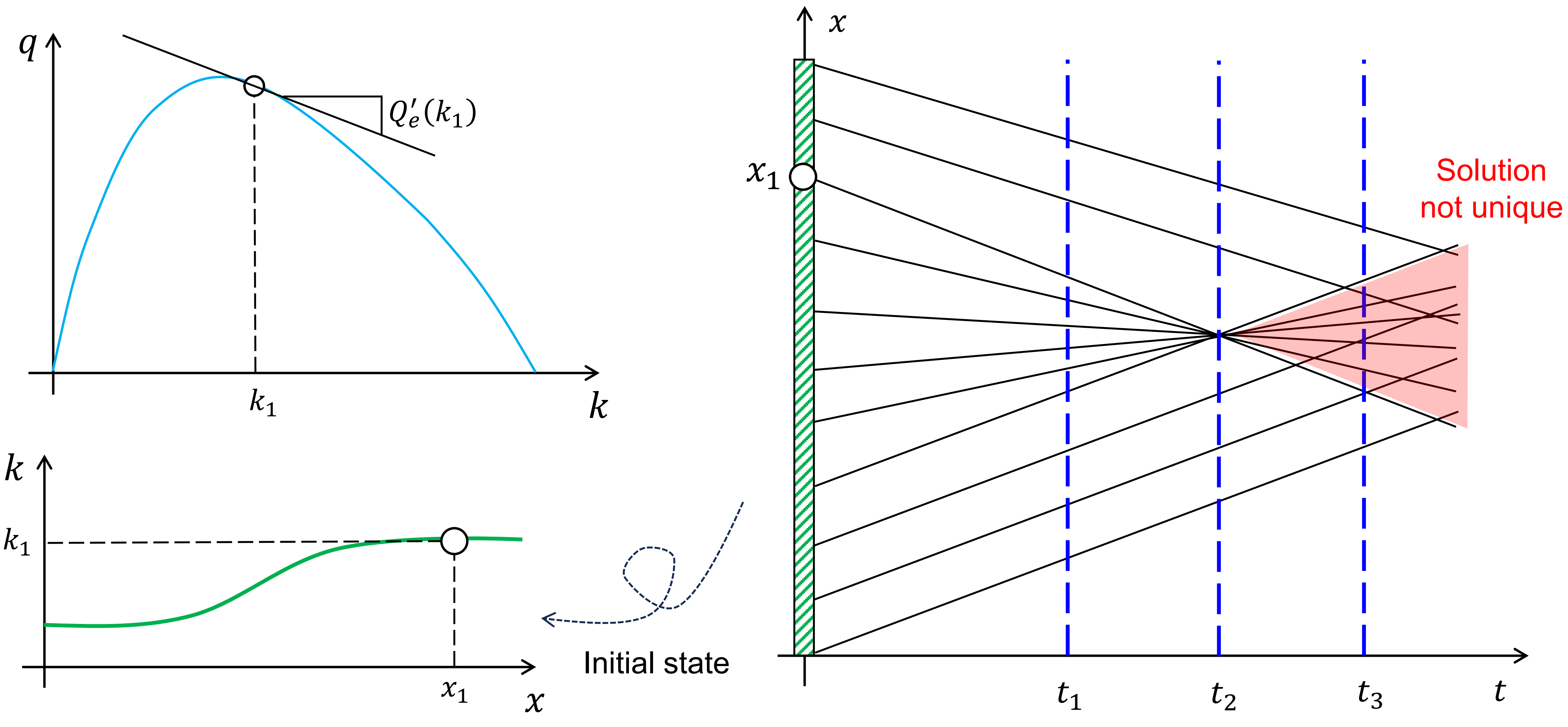


Solving LWR model with characteristic line



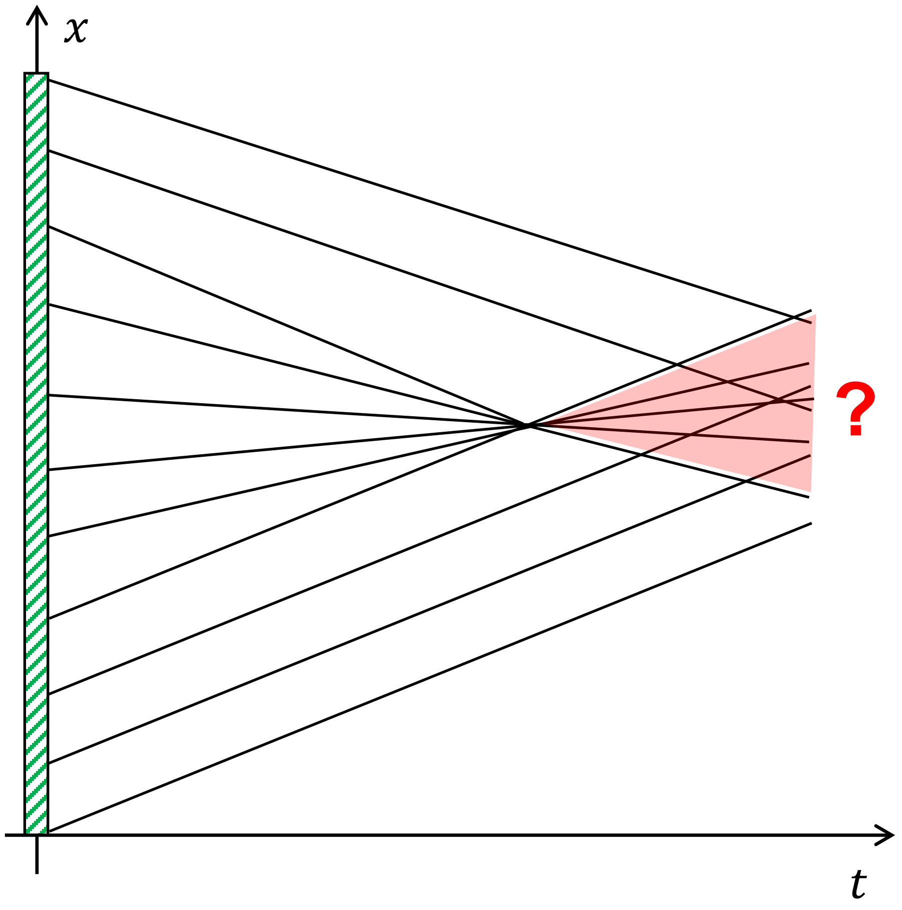


Solving LWR model with characteristic line



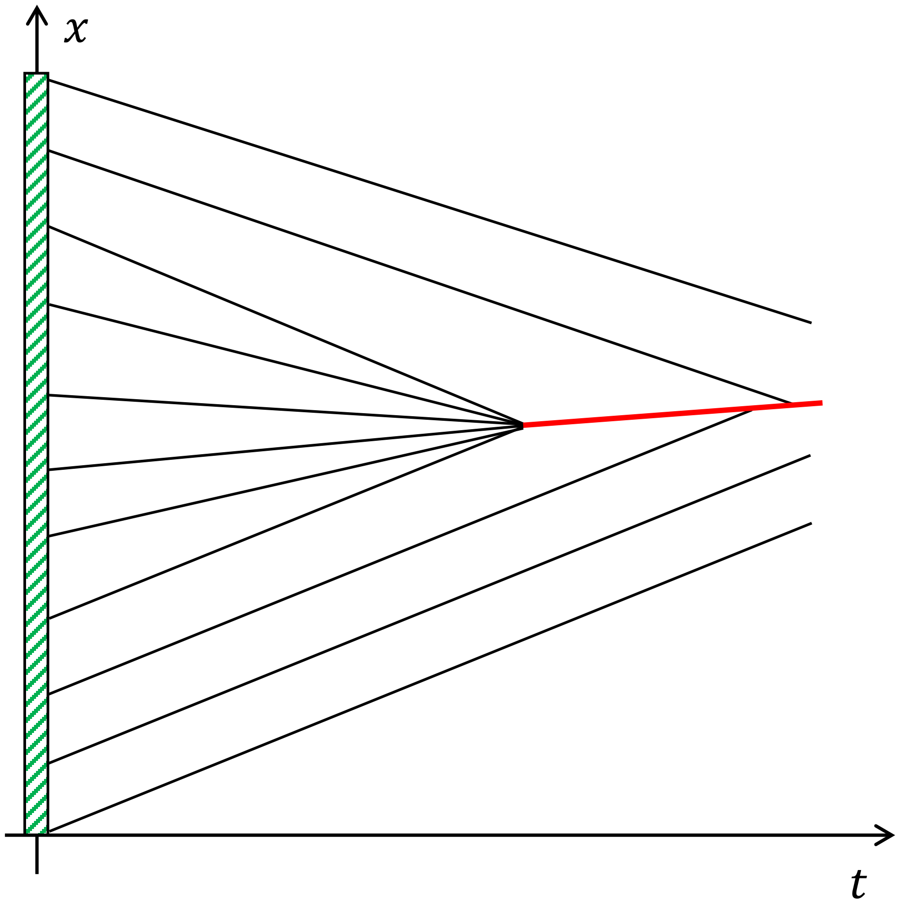
Shockwave

- What happens when characteristic lines intersect with each other?

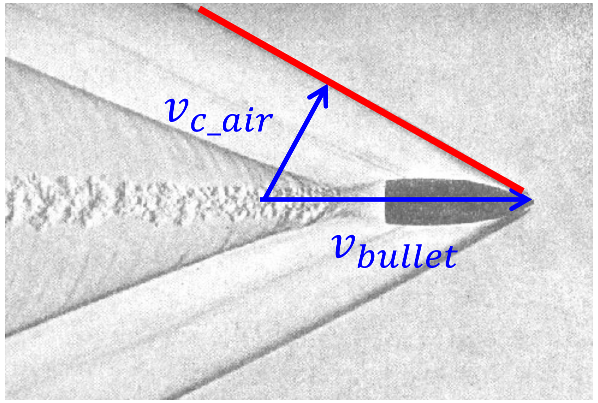


Discontinuity and shockwave

- What happens when characteristic lines intersect with each other?



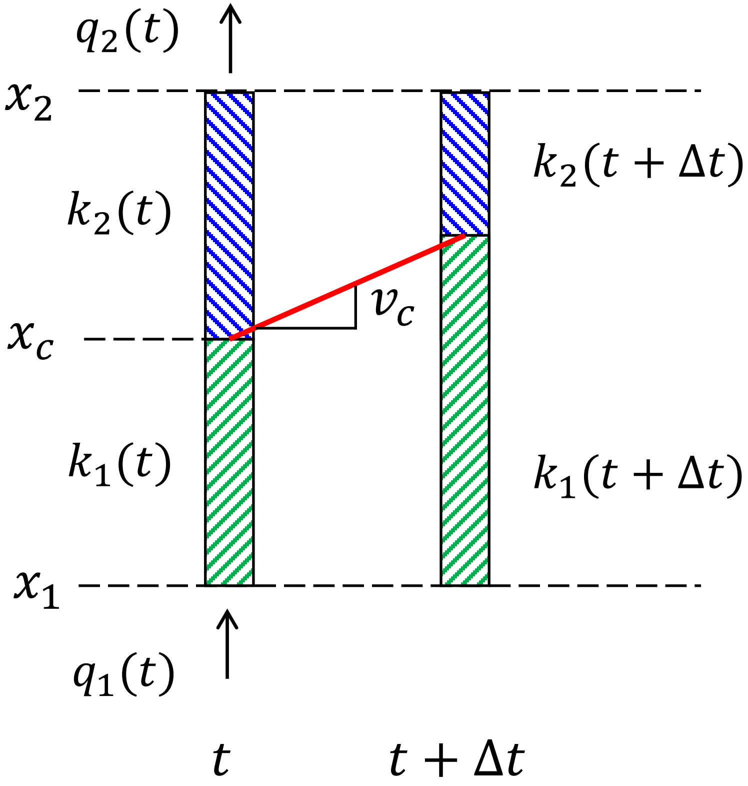
- When characteristic lines intersect with each other, the density $k(x,t)$ is no longer continuous, there will be a jump across certain boundary (discontinuity)
- This discontinuity boundary is called ***shockwave***



- What is in common: shockwave is the boundary between continuous states (there is a sharp change across the shockwave)

Shockwave speed

- The speed of the shockwave is constrained by the conservation law



- We focus on the neighbor of the shockwave line, i.e., $x_1 < x_c < x_2,\ x_2 - x_1 = \Delta x \to 0$
- According to the conservation law

$$k_1(t+\Delta t)(x_c + \Delta x_c - x_1) + k_2(t+\Delta t)(x_2 - x_c - \Delta x_c)$$
$$= k_1(t)(x_c - x_1) + k_2(x_2 - x_c) + q_1(t)\Delta t - q_2(t)\Delta t$$

⇩ (ignore higher-order small quantities, see detailed derivation in the next page)

$$k_1(t)\Delta x_c - k_2(t)\Delta x_c = q_1(t)\Delta t - q_2(t)\Delta t$$

$$\Rightarrow \quad \frac{\Delta x_c}{\Delta t} = \frac{q_2(t) - q_1(t)}{k_1(t) - k_1(t)} = v_s$$ Shockwave speed

Derivation of shockwave speed

$$q_1(t)\Delta t - q_2(t)\Delta t = \underbrace{k_1(t+\Delta t)(x_c+\Delta x_c-x_1) + k_2(t+\Delta t)(x_2-x_c-\Delta x_c) - k_1(t)(x_c-x_1) - k_2(x_2-x_c)}_{= k_1(t)\Delta x_c - k_2(t)\Delta x_c}$$

$$k_1(t+\Delta t)(x_c+\Delta x_c-x_1)$$

$$= \big(k_1(t+\Delta t) + k_1(t) - k_1(t)\big)\cdot(x_c+\Delta x_c-x_1)$$

$$= \left(\frac{dk_1(t)}{dt}\Delta t + k_1(t)\right)\cdot(x_c+\Delta x_c-x_1)$$

$$= \frac{dk_1(t)}{dt}\Delta t\cdot(x_c+\Delta x_c-x_1) + k_1(t)\cdot(x_c+\Delta x_c-x_1)$$

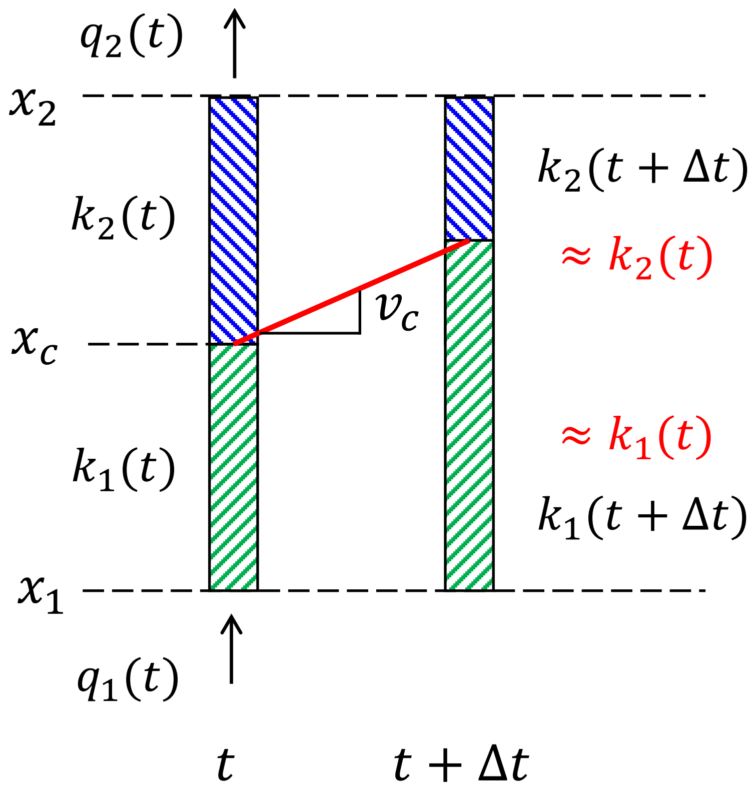
$$\approx k_1(t)\cdot(x_c+\Delta x_c-x_1)$$

Since $(x_c+\Delta x_c-x_1) < x_2 - x_1 = \Delta x$

Similarly, we have: $k_2(t+\Delta t)(x_2-x_c-\Delta x_c) \approx k_2(t)(x_2-x_c-\Delta x_c)$

$$\Rightarrow \quad \frac{\Delta x_c}{\Delta t} = \frac{q_2(t)-q_1(t)}{k_1(t)-k_1(t)} = v_s$$

Intuition of the derivation



- Number of vehicles at time t

$$k_1(t)\cdot(x_c-x_1)+k_2(t)\cdot(x_2-x_c)$$

- Number of vehicles at time $t+\Delta t$

$$k_1(t+\Delta t)\cdot(x_c-x_1+\Delta x_c)+k_2(t+\Delta t)\cdot(x_2-x_c-\Delta x_c)$$

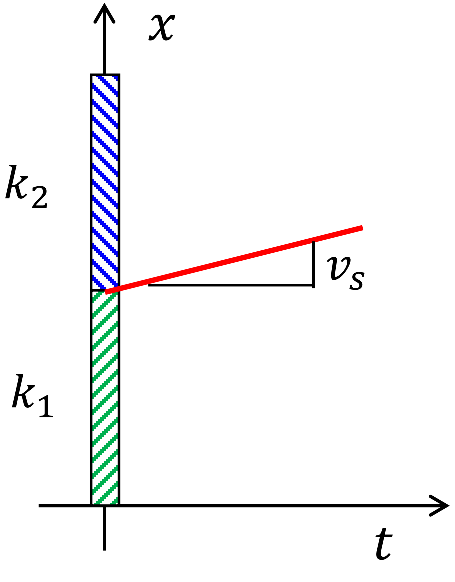
- Two main causes of the changes
  - ~~Change of the densities (both $k_1(t)$ and $k_2(t)$) from time t to $t+\Delta t$~~ (higher-order small term)
  - Change of the boundary

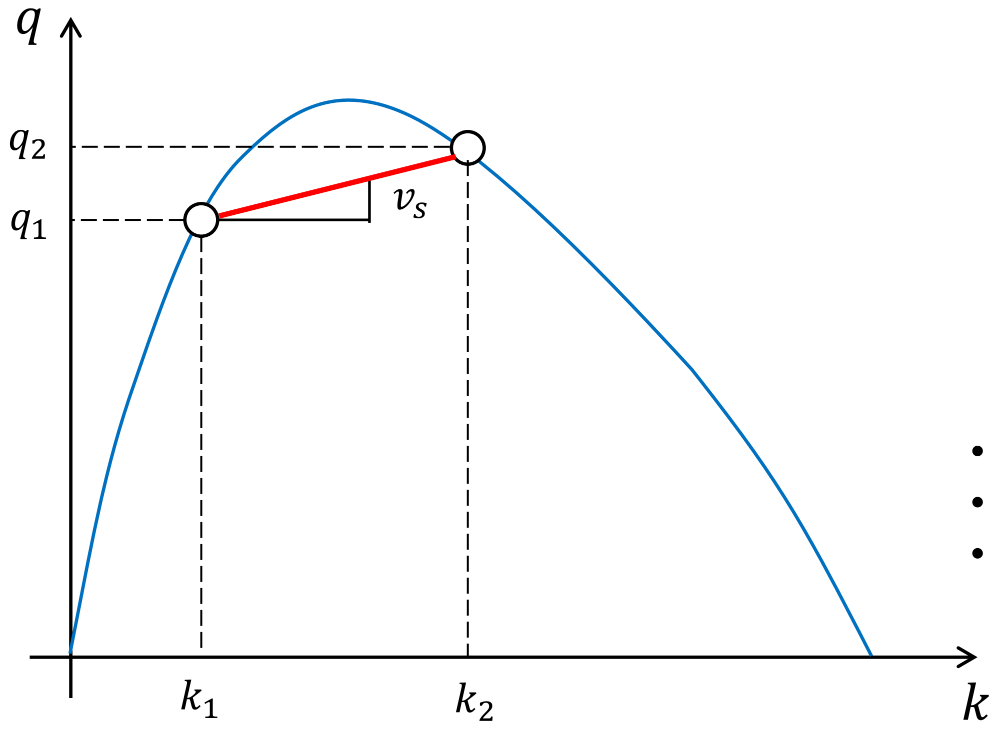
- Final change of number of vehicles

$$k_1(t)\Delta x_c-k_2(t)\Delta x_c$$

Shockwave speed

- Shockwave speed is the slope of the line connecting two traffic states in the fundamental diagram



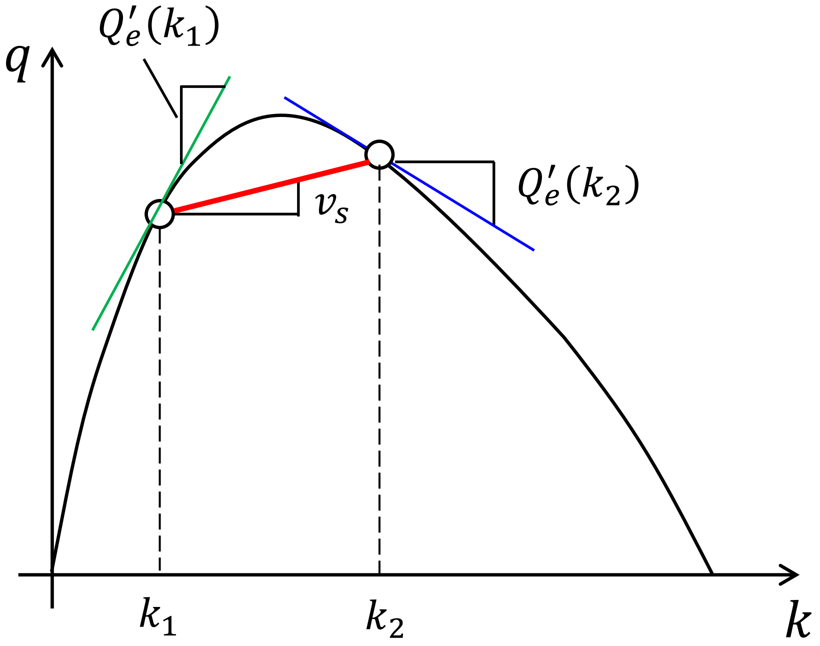


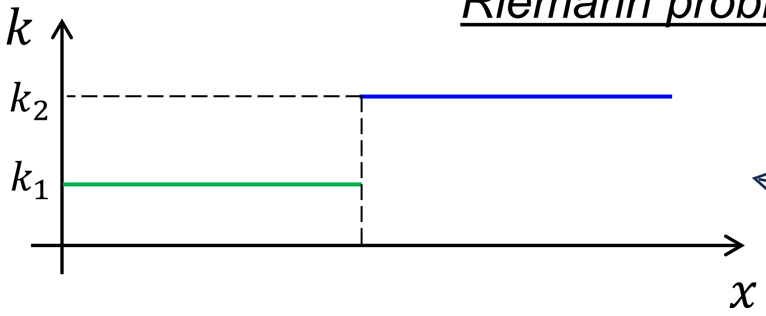
Shockwave speed

$$v_s = \frac{q_2 - q_1}{k_2 - k_1}$$

- $v_s > 0$: moving to the ***downstream***
- $v_s = 0$: stationary
- $v_s < 0$: moving to the ***upstream***

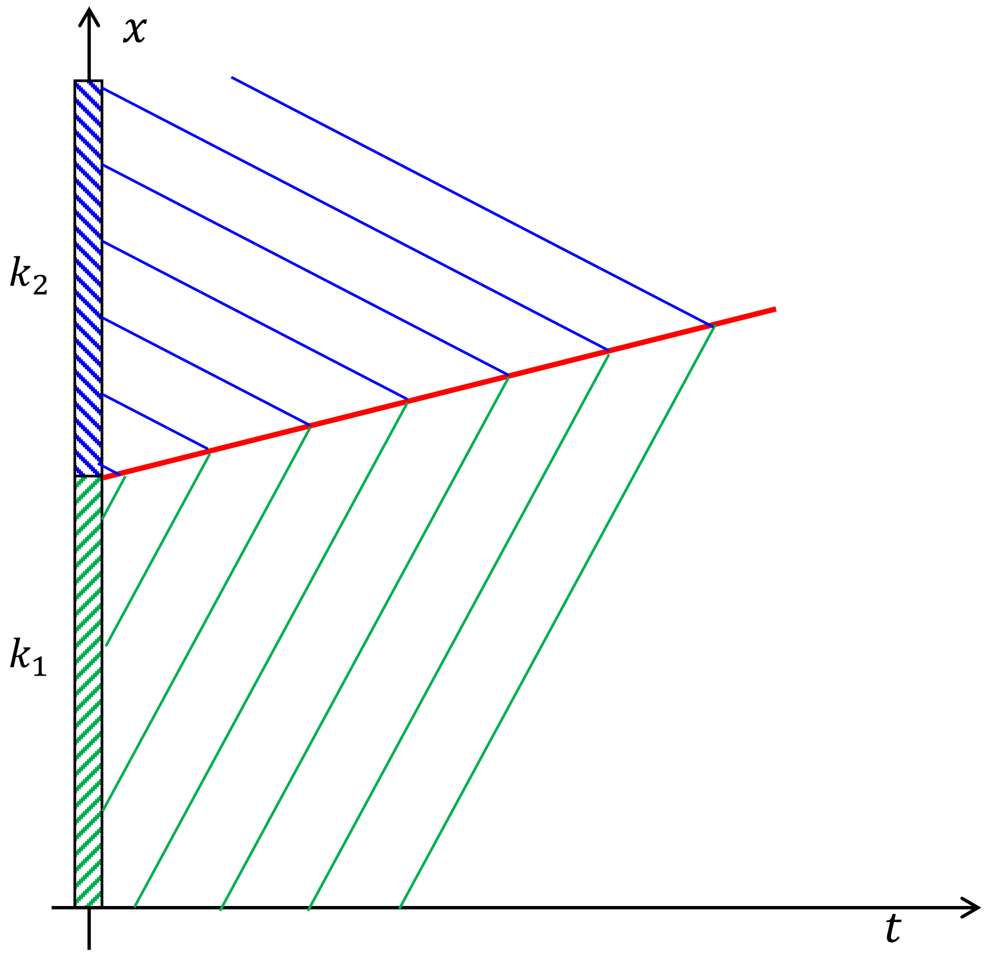
Shockwave solution



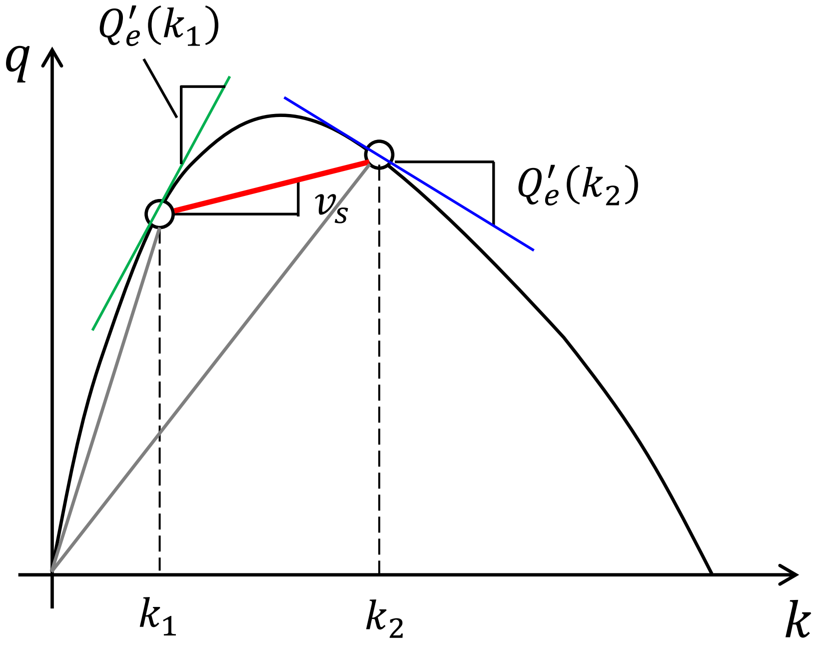


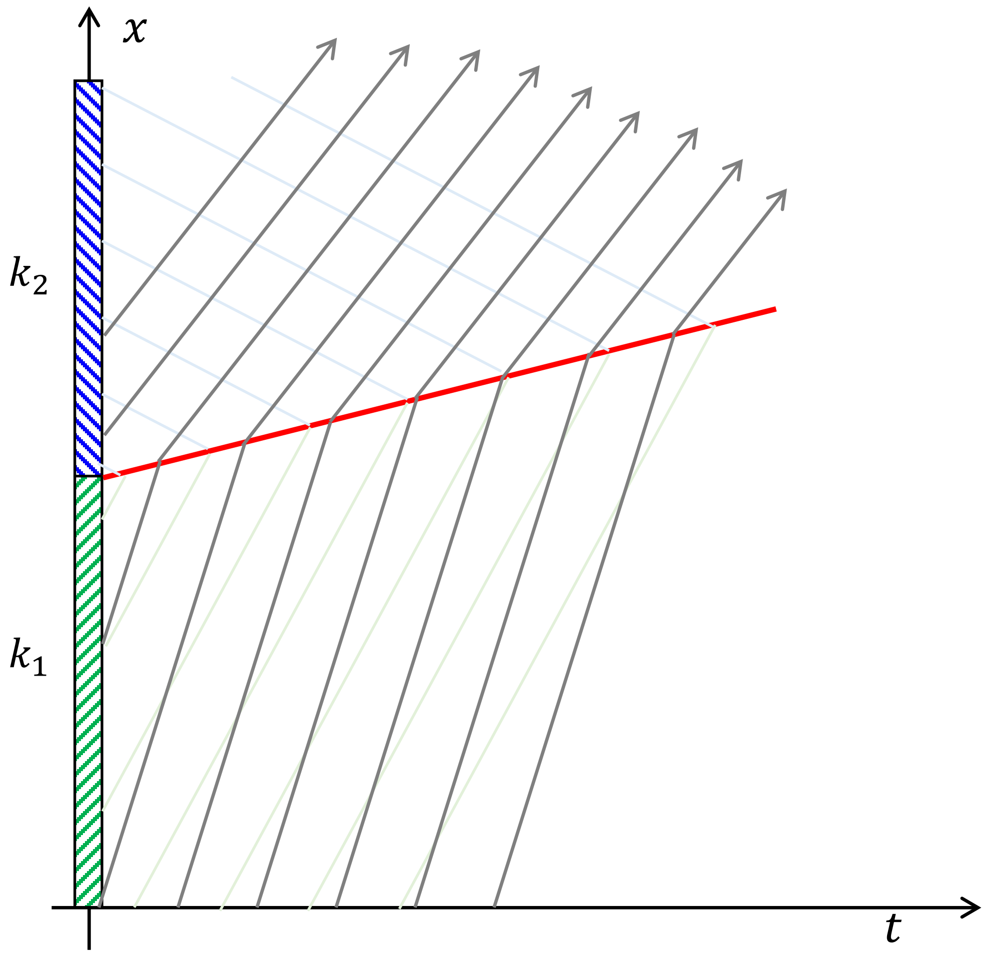
Riemann problem

Initial state

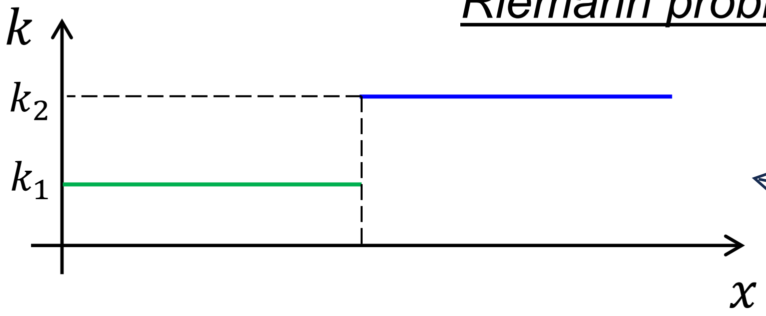


What should trajectories look like?





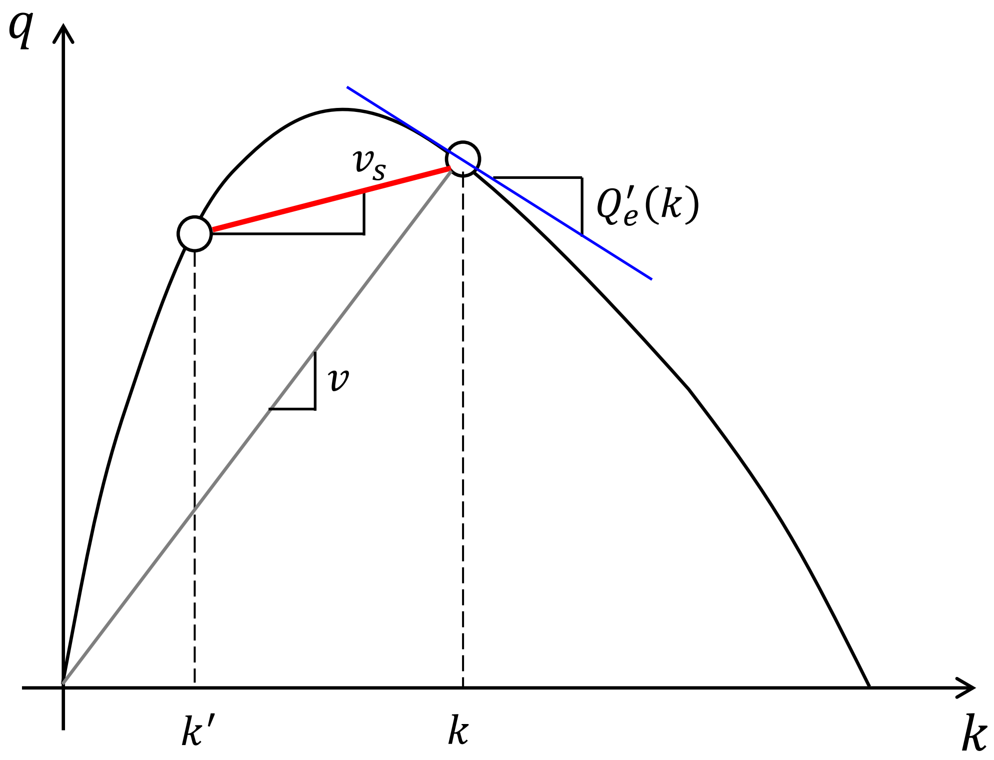
Riemann problem



Initial state

Vehicle speed, shockwave, and characteristic line

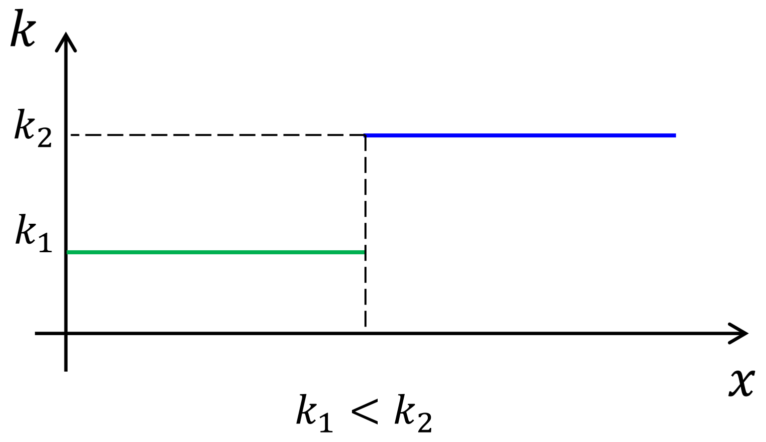
- These three have different meanings, definitions, and speeds

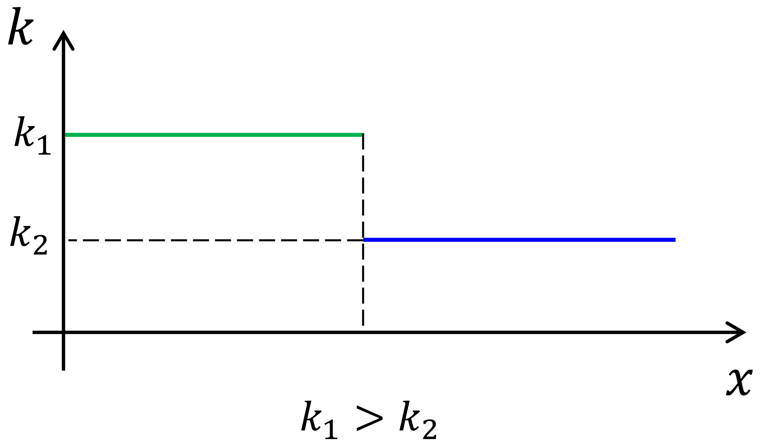


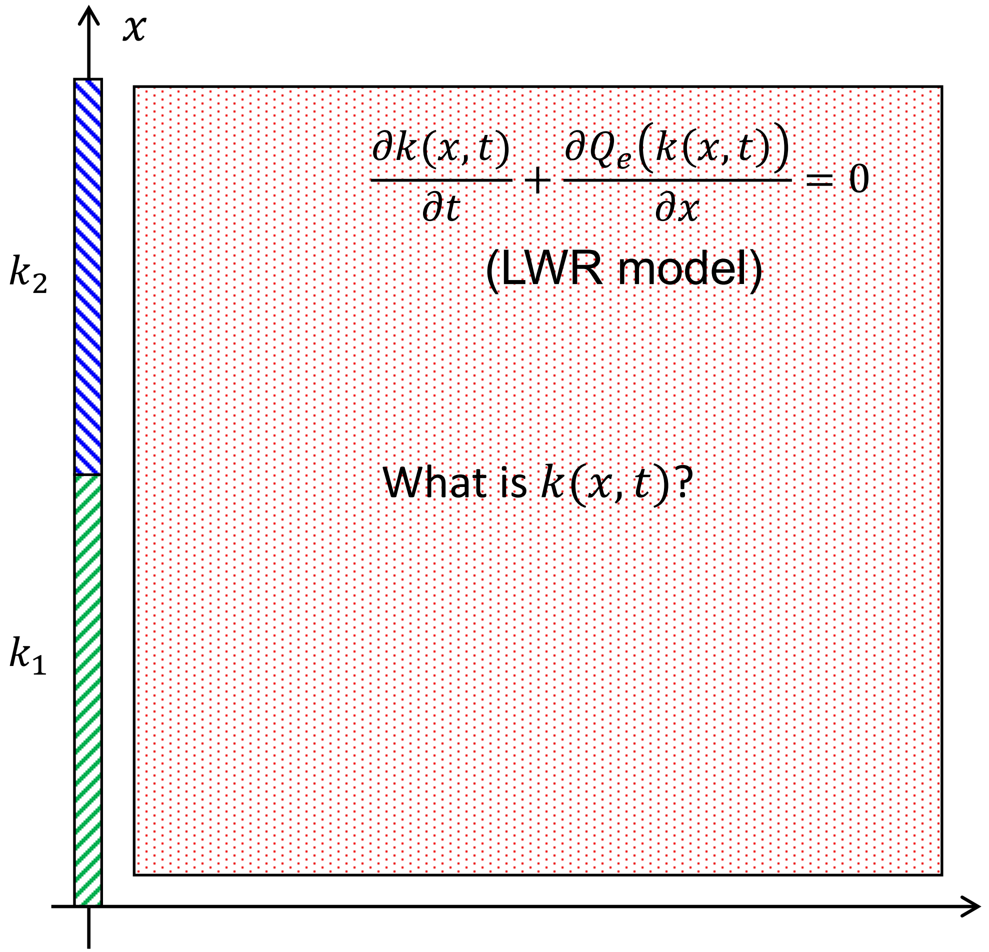
Name	Space-mean speed	Characteristic line	Shockwave
Meaning	Average vehicle traveling speed at certain time and location	Traffic density stays the same along the characteristic line (when solving LWR model)	Discontinuity boundary between two traffic states
Notation	$v = \frac{q}{k}$	$Q'_e(k)$	$v_s = \frac{q - q'}{k - k'}$

Riemann problem

- Riemann problem: solving the LWR model with step initial state

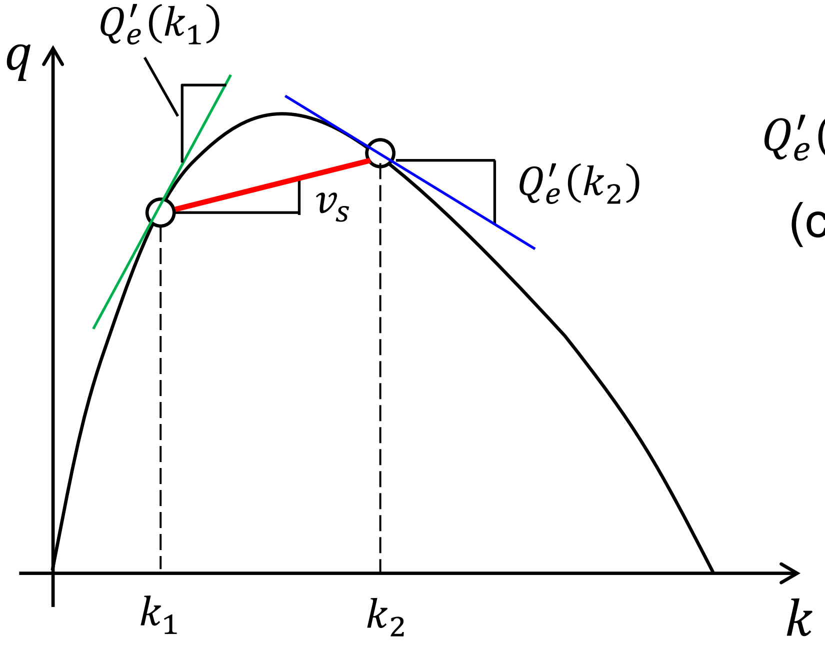






Riemann problem: shockwave solution

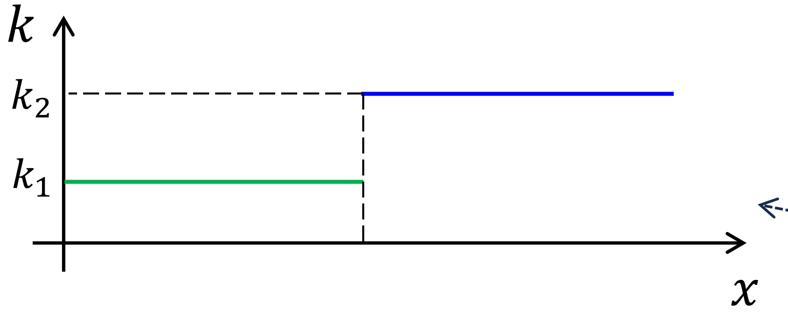
- We will get a shockwave solution if $k_1 < k_2$



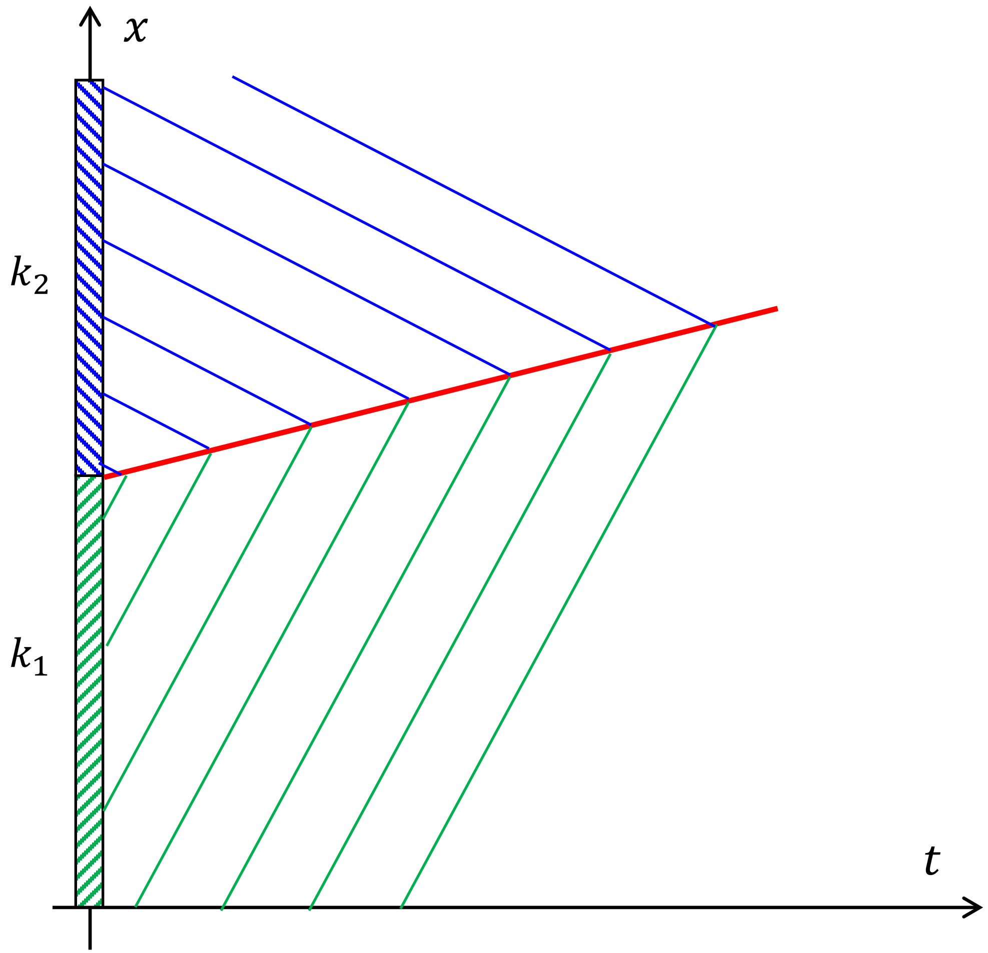
$k_1 < k_2$

$Q_e'(k_1) > Q_e'(k_1)$

(concave FD)

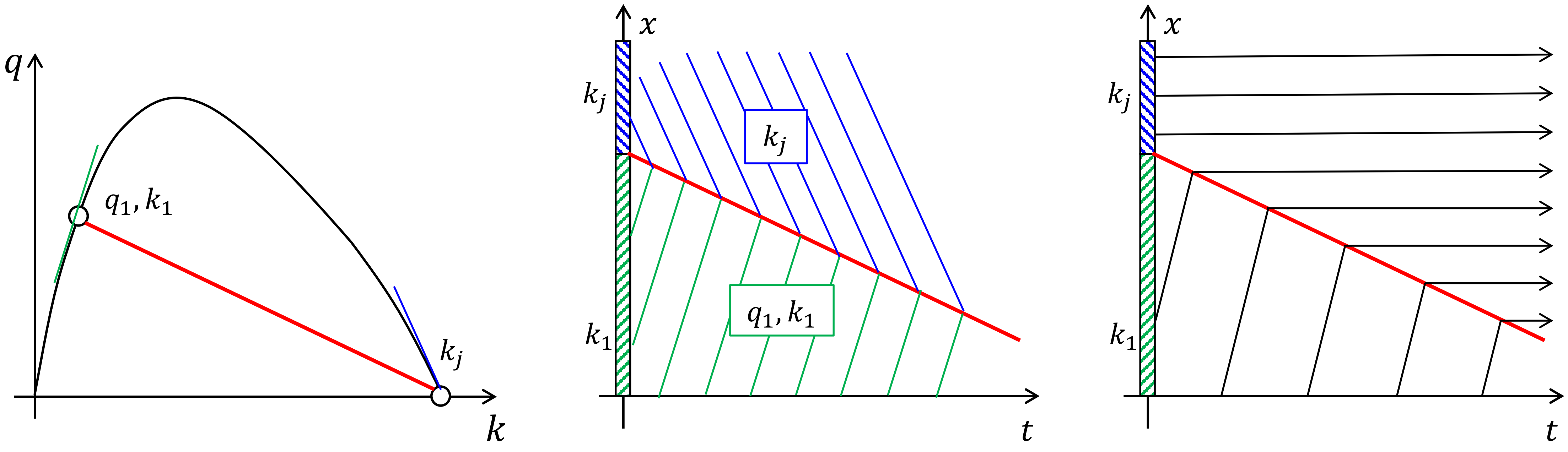


Initial state



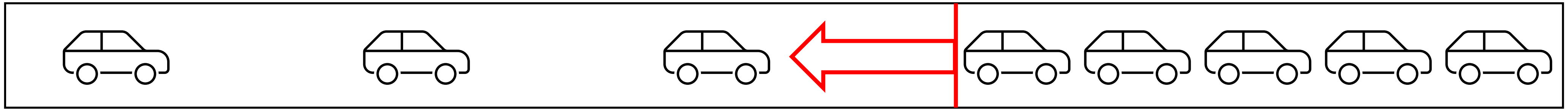
Shockwave example: queue build-up

- We have upstream constant arrival and downstream queue (jam density)



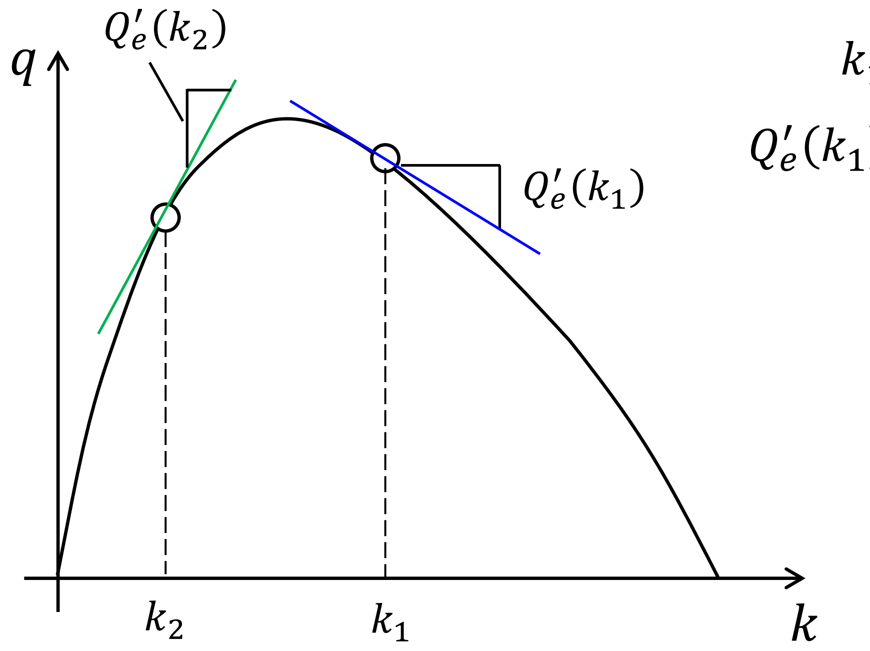
(Characteristic lines & shockwave)

(Vehicle trajectories & shockwave)



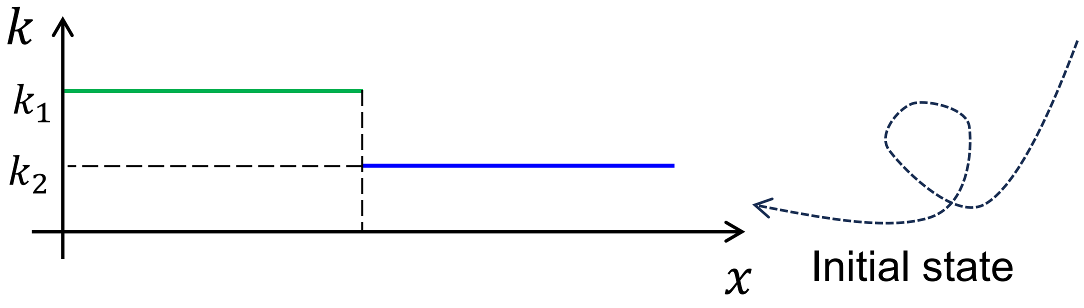
What if $k_1 > k_2$?

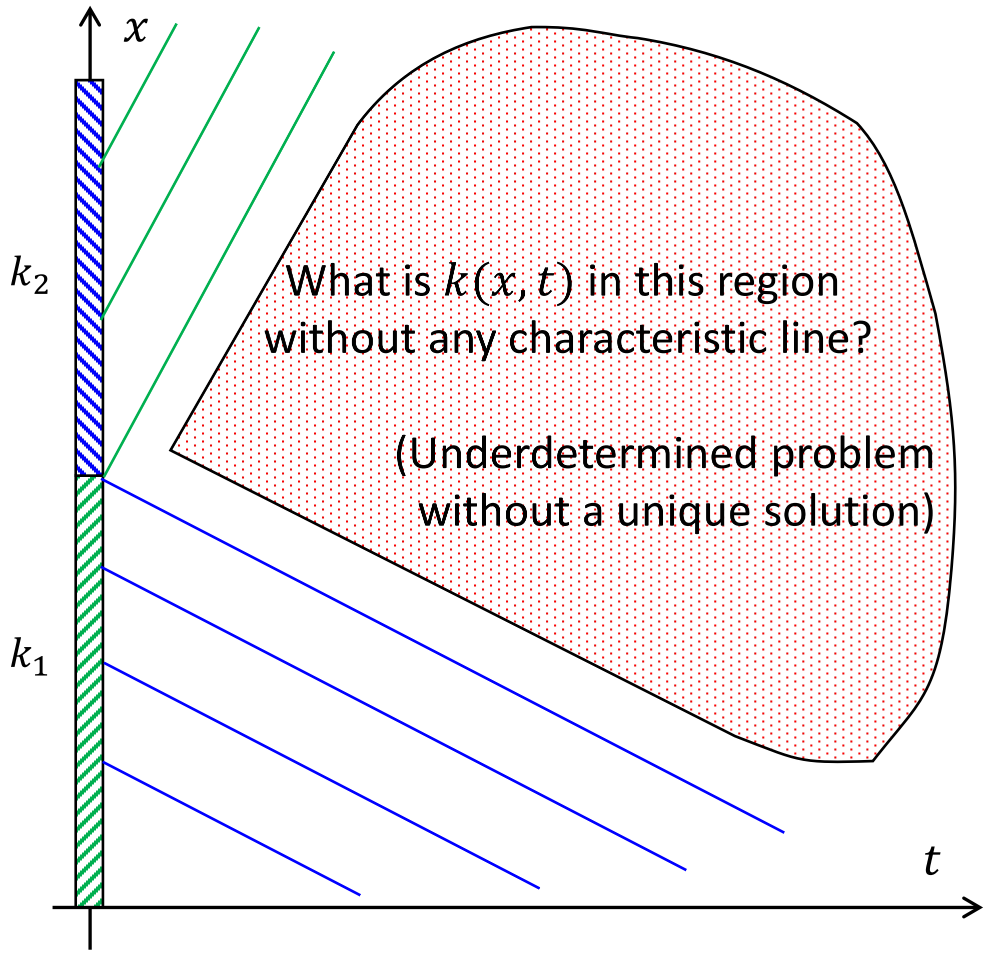
- The characteristic line will not intersect with each other if $k_1 > k_2$



$k_1 > k_2$

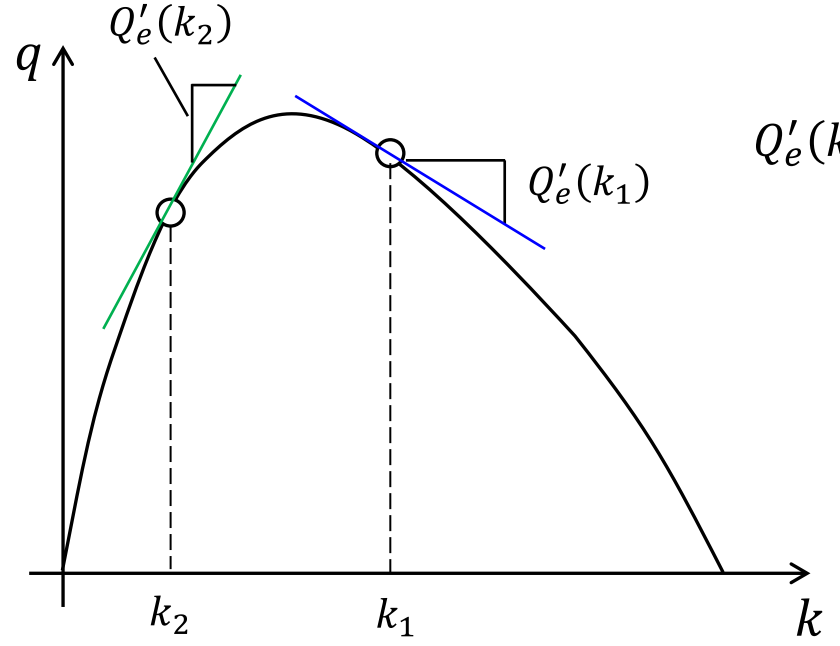
$Q_e'(k_1) < Q_e'(k_1)$





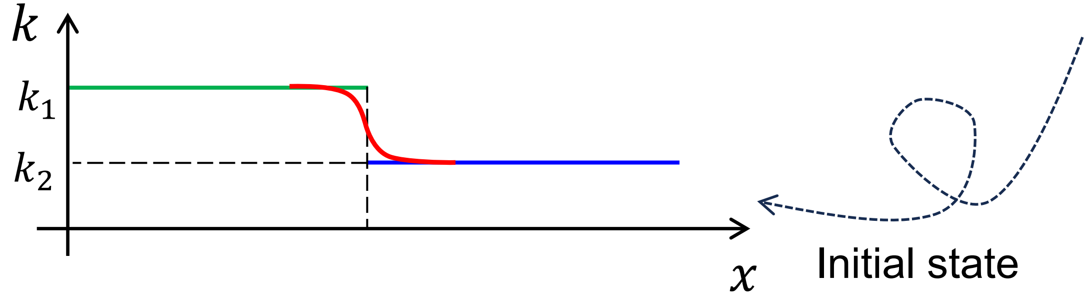
Riemann problem: rarefaction fan solution

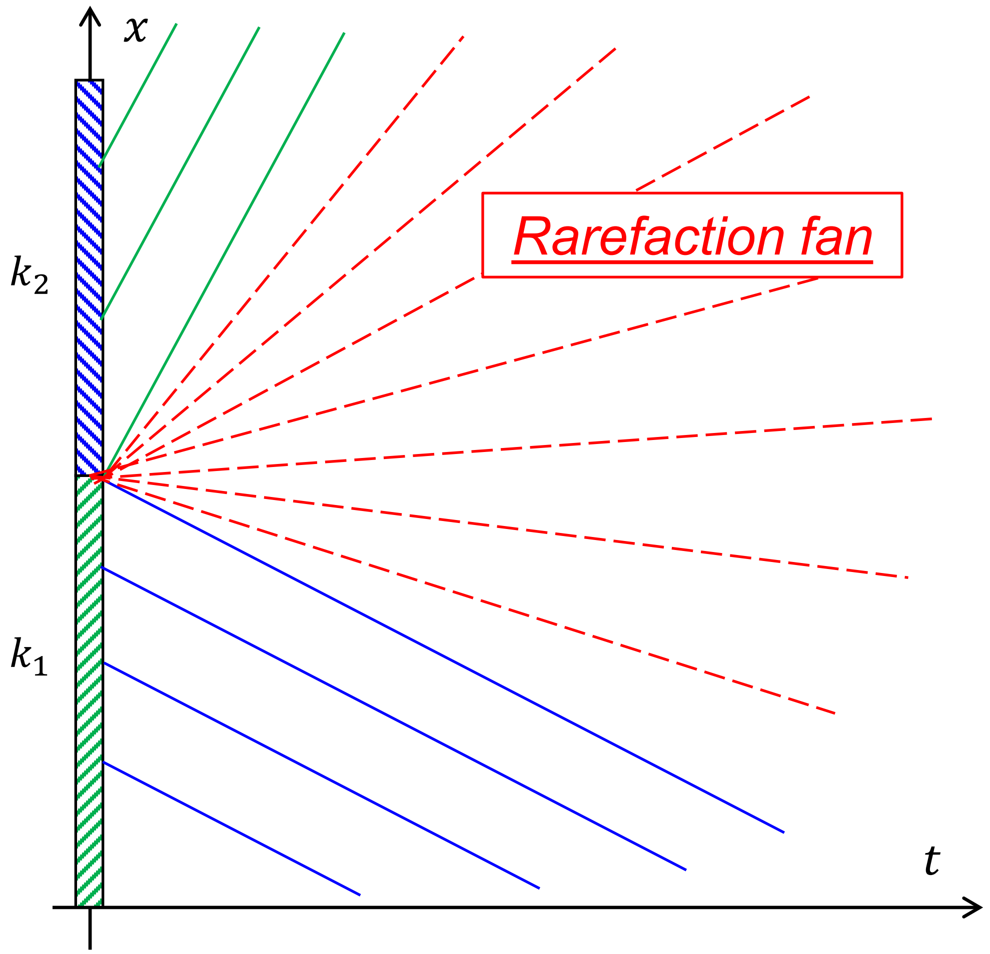
- We can twist the input step function into a continuous function



$k_1 > k_2$

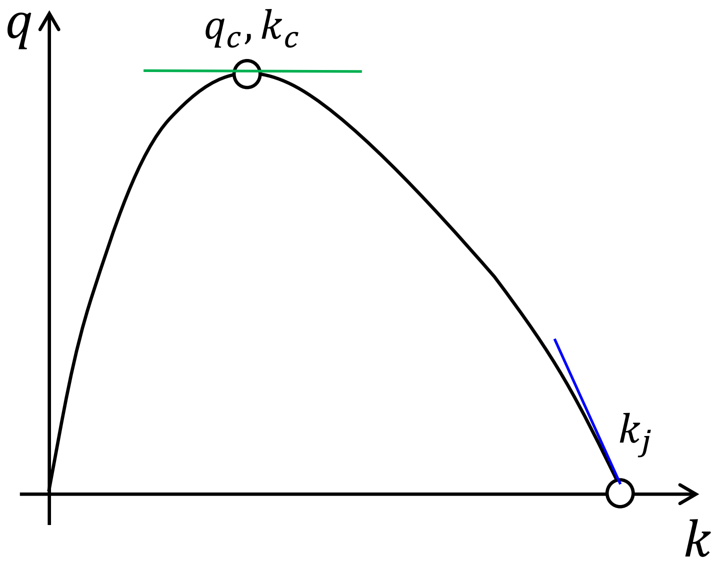
$Q'_e(k_1) < Q'_e(k_1)$

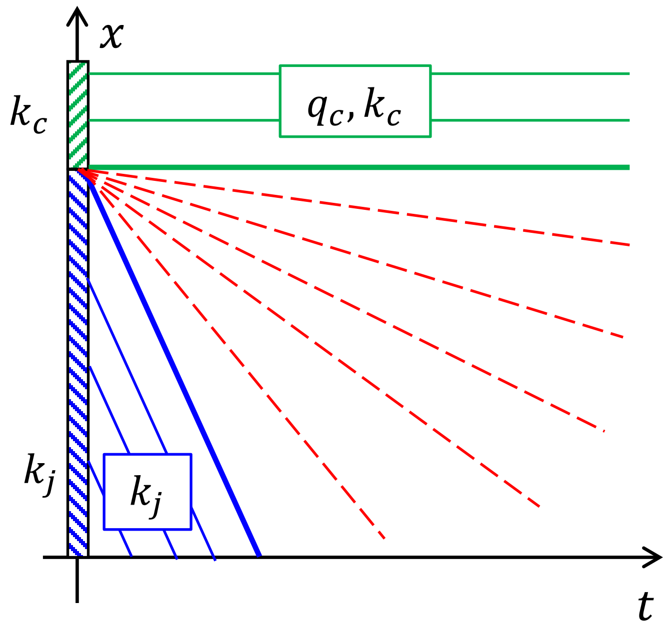




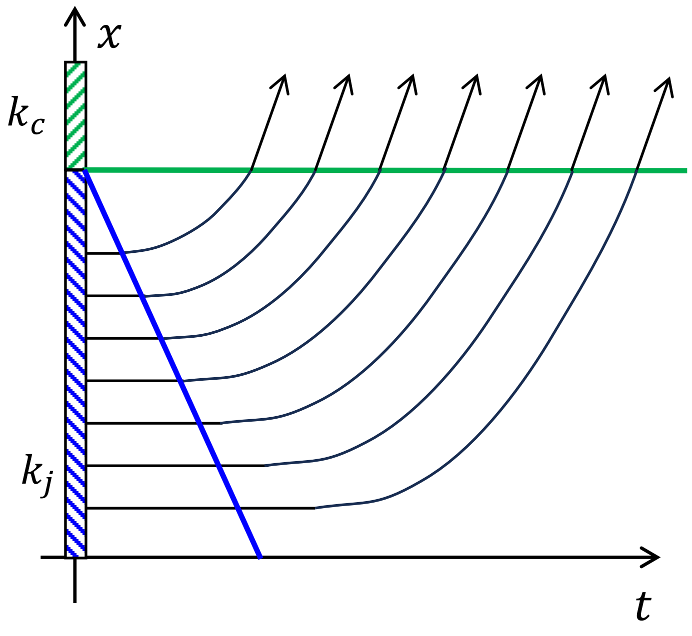
Rarefaction example: queue dissipation

- Vehicle discharging from jam density

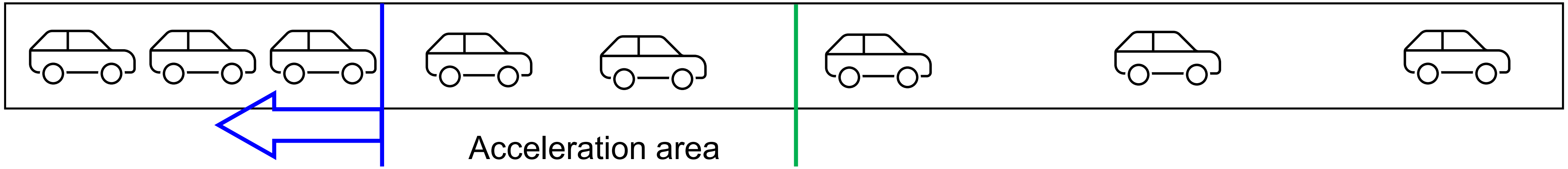




(Characteristic lines & shockwave)



(Vehicle trajectories & shockwave)



Some suggestions

- This lecture has extensive mathematical derivations. You only need to know the intuition behind it
- It is normal that you do not fully understand during this lecture, I suggest you work on your own after the class
- There will be no homework or exam regarding the mathematical derivations
- It is even OK you cannot understand the derivation

- The next lecture will summarize the solution to the LWR model, which should be easier to understand

Reading

- Section 5 in TFT_Document.pdf