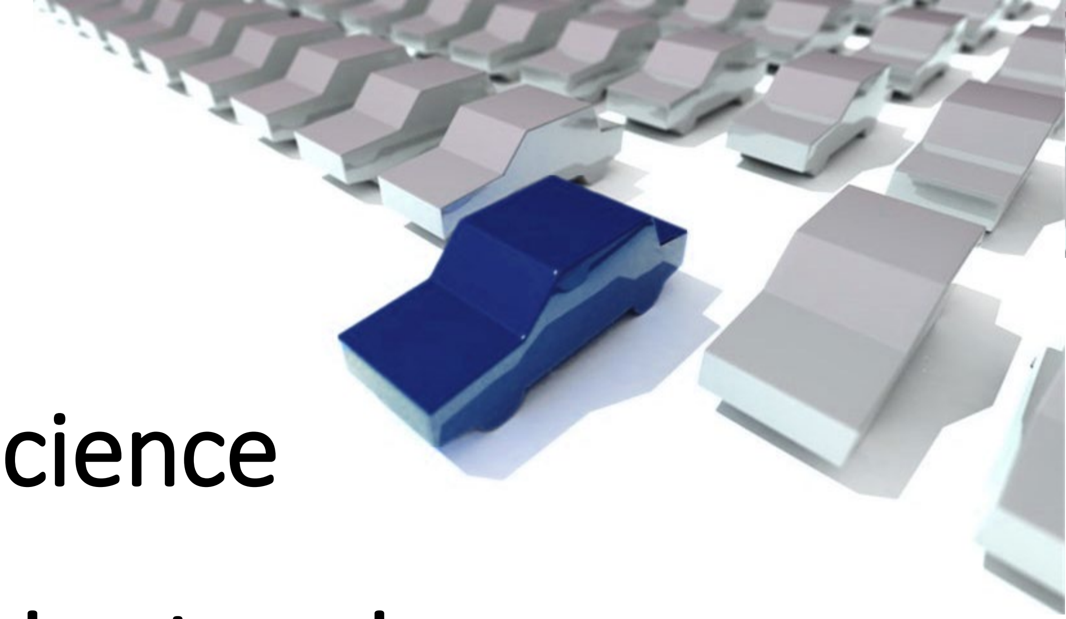


CEE 551 Traffic Science

Traffic Flow Theory Lecture I

Traffic flow variables, fundamental diagram, and conservation law

Dr. Xingmin Wang
Department of Civil and Environmental Engineering
University of Michigan
Email: xingminw@umich.edu

My suggestions on learning traffic flow theory

- Traffic flow theory: simple intuition but complicated math. It is the basics of this whole course but the most difficult part
- There will be many mathematical derivations… (particularly the next lecture)
  - At least once, go through and understand the derivation step by step on your own
  - Link the math with physical meaning
  - The ultimate goal is to understand the underlying intuition
- Lectures closely related to each other: if you miss one, you will be lost in the following lectures
- If possible, attend class in person
- Learn with simple examples (if not included in the slides, think about your own simple real-world examples)
- Recommended materials: slides + TFT_document.pdf
- Schedule office hour with me if necessary (email: xingminw@umich.edu)

Outline

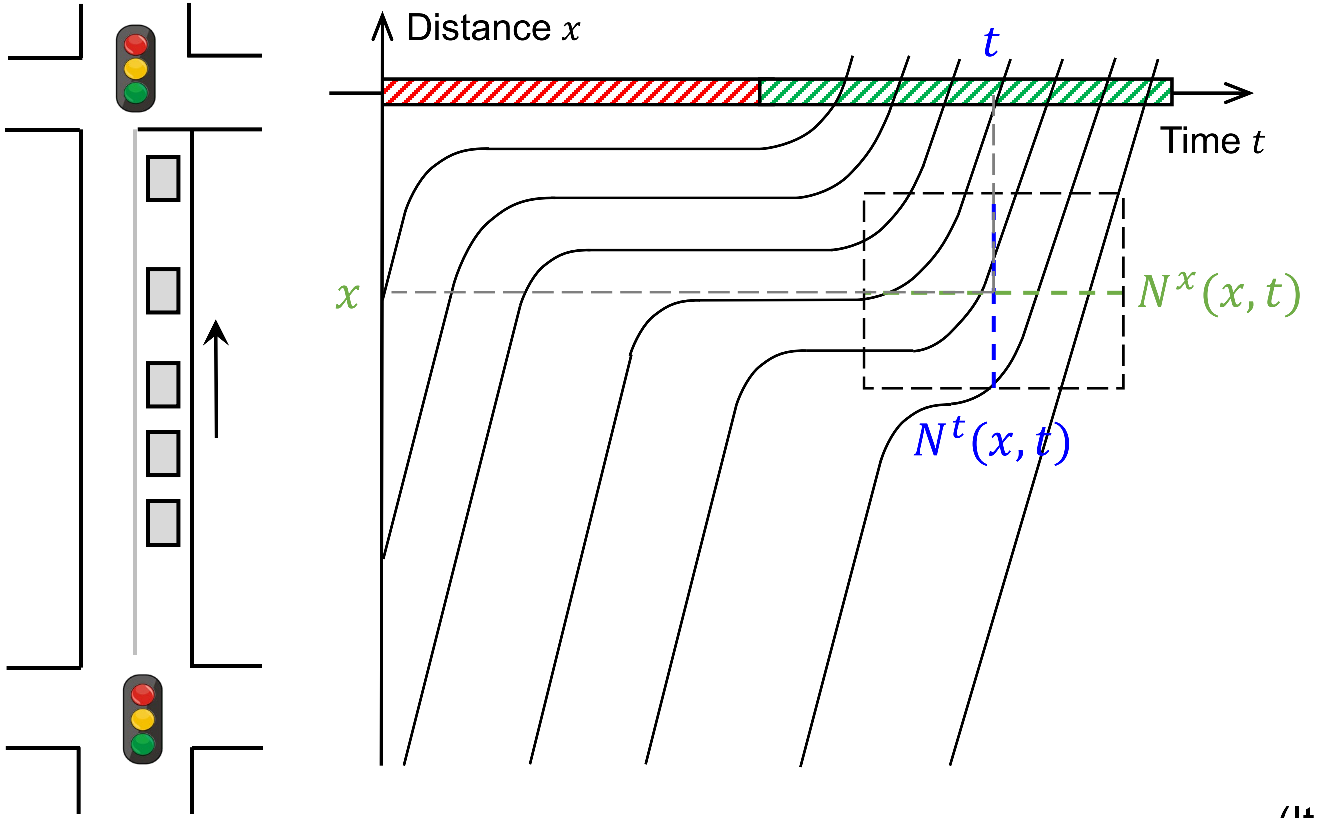
- ❑ Traffic flow variables
- ❑ Measurement of traffic states
- ❑ Fundamental diagram
- ❑ Conservation law

Outline

- ❑ Traffic flow variables
- ❑ Measurement of traffic states
- ❑ Fundamental diagram
- ❑ Conservation law

Time-space diagram, traffic flow and density

- Definition of (average) traffic flow and density based on the time-space diagram



- $N^x(x,t)$: number of vehicles passing location x from $t-\frac{1}{2}\Delta t$ to $t+\frac{1}{2}\Delta t$
- $N^t(x,t)$: number of vehicles traversing time t from $x-\frac{1}{2}\Delta x$ to $x+\frac{1}{2}\Delta x$

- Average traffic density

$$\bar{k}(x,t)=\frac{N^t(x,t)}{\Delta x}\quad\left[\frac{veh}{distance}\right]$$

- Average traffic flow rate

$$\bar{q}(x,t)=\frac{N^x(x,t)}{\Delta t}\quad\left[\frac{veh}{time}\right]$$

(It does not matter it is either from $t-\frac{1}{2}\Delta t$ to $t+\frac{1}{2}\Delta t$ or from t to $t+\Delta t$ when $\Delta t\rightarrow 0$, the same for x)

Space-mean speed and time-mean speed

- Space-mean speed (SMS, more frequently used)
  - Defined based on a certain road segment $[x, x+\Delta x]$ and time interval $[t, t+\Delta t]$
  - Individual vehicle's travel time τ_i is measured
  - Distance traveled divided by the average travel time

$$\bar{v}^s(x,t) = \frac{\Delta x}{\frac{1}{N}\sum_{i=1}^{N}\tau_i} = \frac{1}{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{v_i}}$$

(Harmonic mean)

  - (Total travel distance divided by total travel time)

Harmonic mean $\leq$ Arithmetic mean

- Time-mean speed (TMS)
  - Measure the speed of individual vehicle as $v_i(t)$
  - Average of individual vehicle speeds at certain time t

SMS $\leq$ TMS

$$\bar{v}^t(t) = \frac{1}{N}\sum_{i}^{N} v_i(t)$$

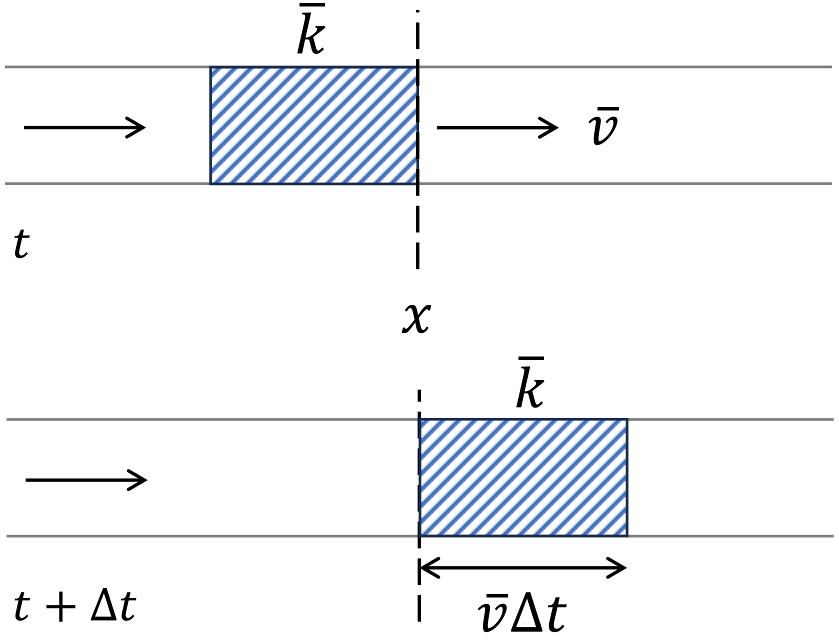
(Arithmetic mean)

Macroscopic traffic flow variables

Name	Notation	Meaning	Units
Average density	$\bar{k}(x,t)$	The average number of vehicles per unit length of the road at position x and time t	$\frac{[veh]}{[distance]}$
Average flow rate	$\bar{q}(x,t)$	The average number of vehicles to cross position x per unit time at time t	$\frac{[veh]}{[time]}$
Average speed	$\bar{v}(x,t)$	The mean speed of vehicles at position x at time t	$\frac{[distance]}{[time]}$

$$\bar{q}(x,t) = \bar{k}(x,t) \cdot \bar{v}(x,t)$$

A concise proof of $\bar{q} = \bar{k} \cdot \bar{v}$



- Number of vehicles passing location x from time t to $t + \Delta t$

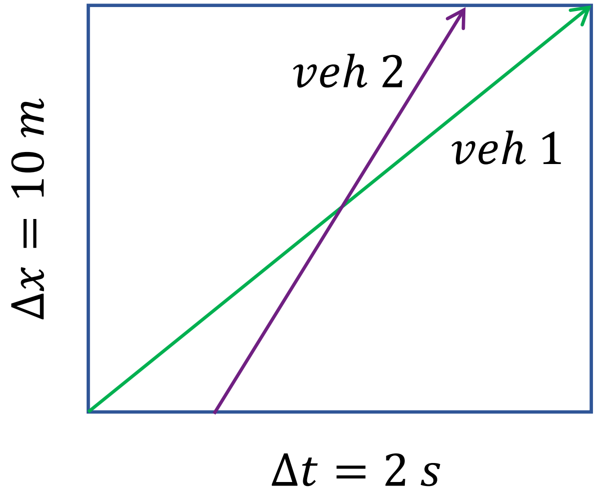
$$\Delta N = \bar{v}(x,t)\Delta t \cdot \bar{k}(x,t)$$

- Flow rate at (x,t):

$$\bar{q}(x,t) = \frac{\Delta N}{\Delta t} = \bar{v}(x,t)\bar{k}(x,t)$$

Example: average speed calculation

- A road segment ($10\ m \times 2\ sec$) with two vehicles passing through with constant speeds ($5m/s$ and $10m/s$)



$\tau_1 = 2\ s, \qquad v_1 = 5\ m/s$

$\tau_2 = 1s, \qquad v_2 = 10\ m/s$

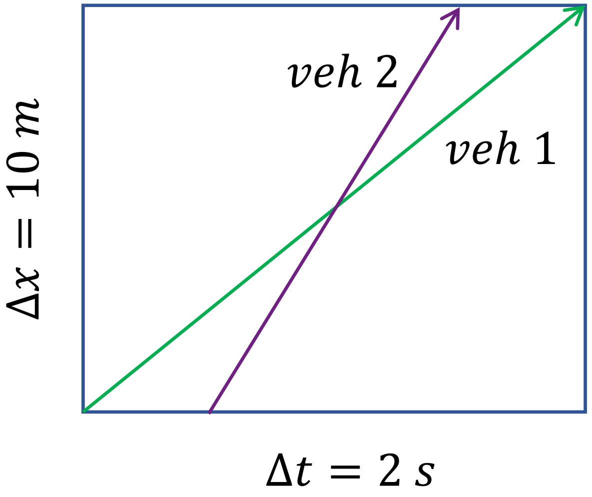
$$\bar{v}^t = \frac{v_1 + v_2}{2} = \frac{5+10}{2} = 7.5\ \ m/s$$

$$\bar{v}^s = \frac{\Delta x}{\frac{1}{2}\cdot(\tau_1+\tau_2)} = \frac{10}{\frac{1}{2}\cdot(2+1)} = \frac{20}{3} \approx 6.67\ m/s$$

- Question: in the equality $\bar{q} = \bar{k}\cdot\bar{v}$, is $\bar{v}$ space-mean speed or time-mean speed?

Example: verification of $\bar{q} = \bar{k} \cdot \bar{v}$

- A road segment ($10\ m \times 2\ sec$) with two vehicles passing through with constant speeds ($5 m/s$ and $10 m/s$)



$\tau_1 = 2\ s, \quad v_1 = 5\ m/s$

$\tau_2 = 1 s, \quad v_2 = 10\ m/s$

$$\bar{v}^t = 7.5\ m/s$$

$$\bar{v}^s = \frac{20}{3} \approx 6.67\ m/s$$

$$\bar{q} = \frac{2\ veh}{\Delta t} = \frac{2}{2} = 1\ veh/s$$

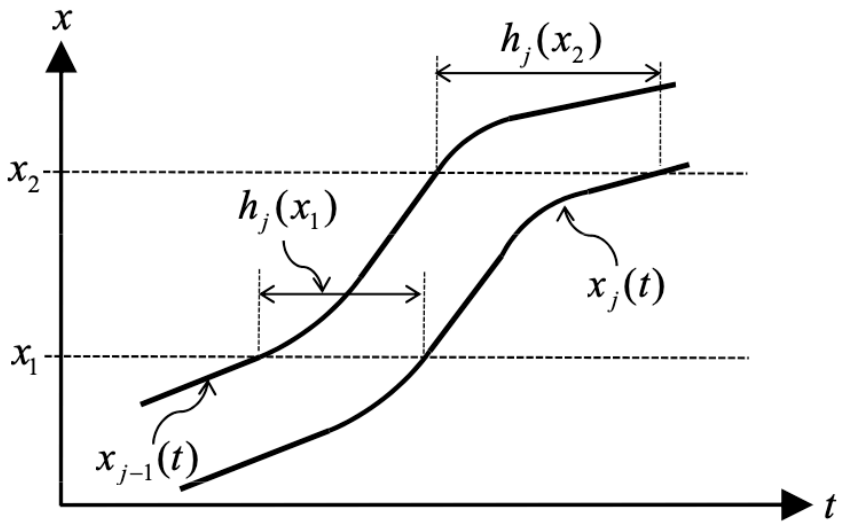
$$\bar{k} = \frac{1}{10\ m} \cdot \frac{2\ veh \cdot 1\ s + 1\ veh \cdot 1\ s}{2\ s} = 0.15\ veh\ /\ m$$

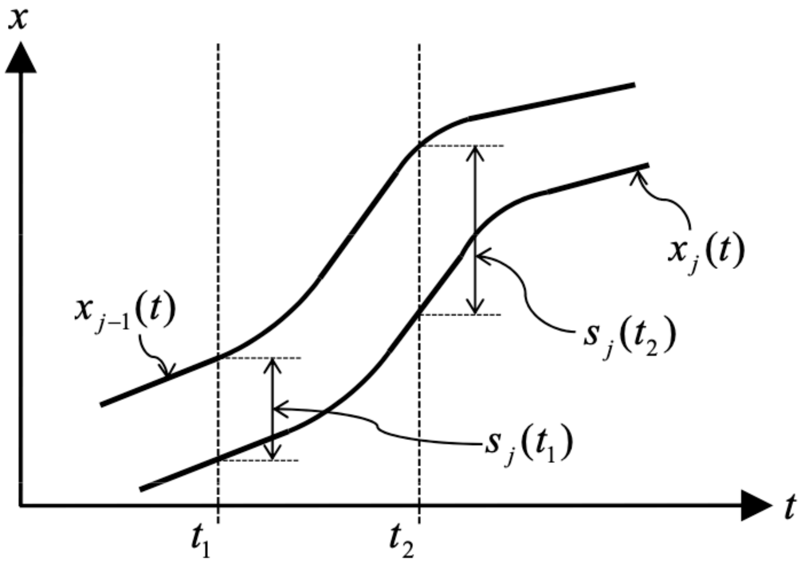
$$\bar{v} = \frac{\bar{q}}{\bar{k}} = \frac{1}{0.15} = \frac{20}{3} = \bar{v}^s$$

- In $\bar{q} = \bar{k} \cdot \bar{v}$, $\bar{v}$ is the space-mean speed

Microscopic traffic flow variables

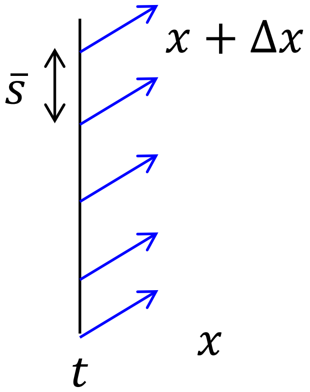
Name	Notation	Meaning	Units
(Time) headway	$h_j(x) = t_j(x) - t_{j-1}(x)$	Time interval between two consecutive crossing times at position x	$\frac{[time]}{[veh]}$
Spacing	$s_j(t) = x_{j-1}(t) - x_j(t)$	Distance between leading and following vehicles at time t	$\frac{[distance]}{[veh]}$





Relationship between different traffic variables

- Average spacing $\bar{s}(x,t)$ and average density $\bar{k}(x,t)$ are reciprocals of each other



Number of vehicles at time t from x to $x + \Delta x$:

$$\Delta N = \frac{\Delta x}{\bar{s}(x,t)}$$

⇒

Average traffic density:

$$\bar{k}(x,t) = \frac{\Delta N}{\Delta x} = \frac{1}{\bar{s}(x,t)}$$

$\bar{k}$: $\left[\frac{veh}{distance}\right]$

$\bar{s}$: $\left[\frac{distance}{veh}\right]$

- Average headway $\bar{h}(x,t)$ and average flow rate $\bar{q}(x,t)$ are reciprocals of each other (with a similar proof)

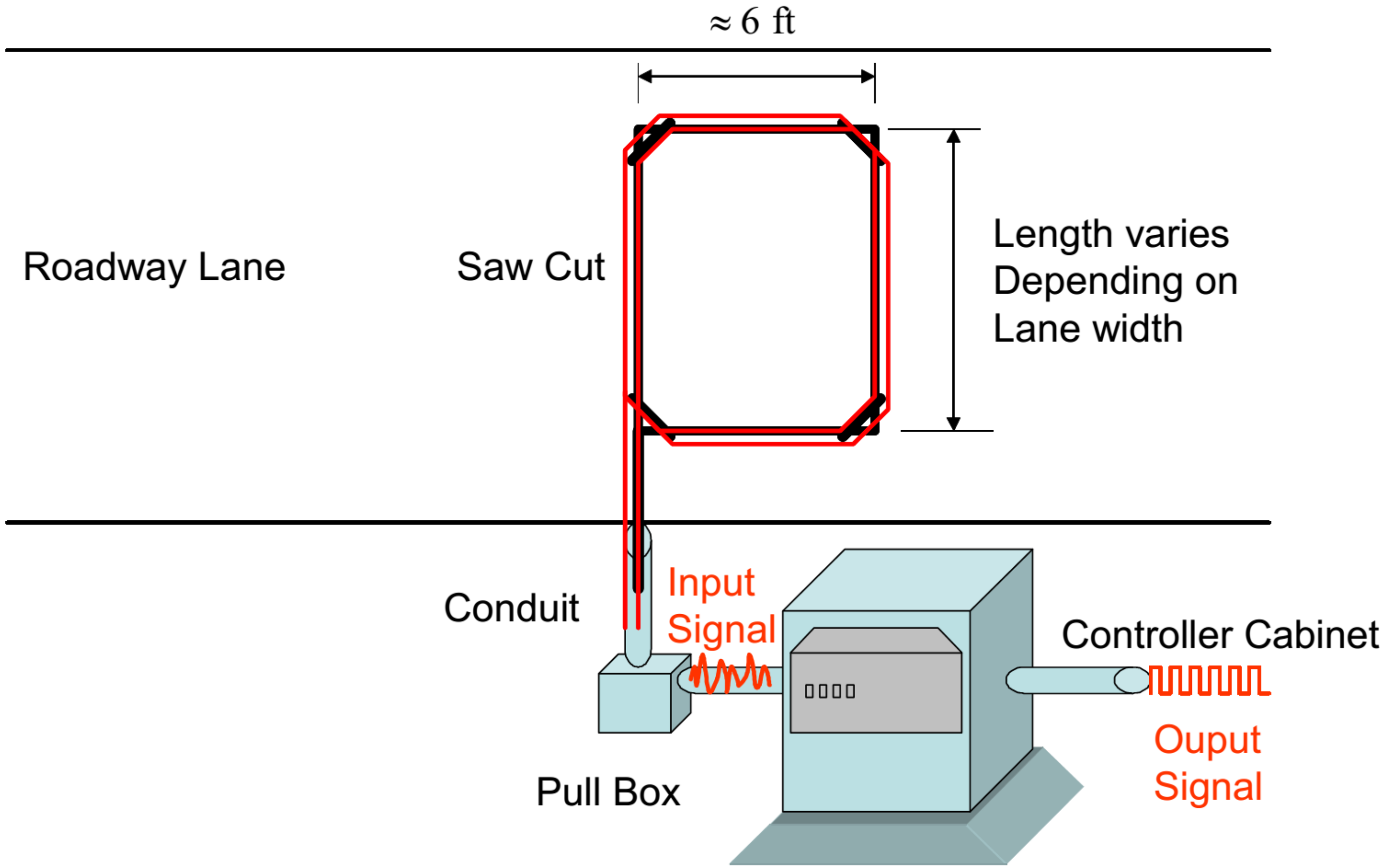
$$\bar{k}(x,t) \cdot \bar{s}(x,t) = 1$$

$$\bar{h}(x,t) \cdot \bar{q}(x,t) = 1$$

Outline

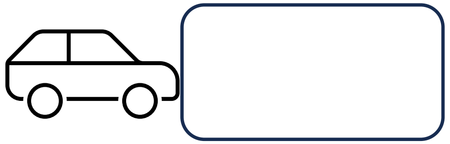
- ❑ Traffic flow variables
- ❑ Measurement of traffic states
- ❑ Fundamental diagram
- ❑ Conservation law

Inductance loop detectors



Traffic flow measurements with loop detectors

- Measurements (outputs) of loop detectors
  - Occupancy (%): proportion of the time with detected vehicles
  - Count (veh): number of vehicles passing through



- For each vehicle, the total travel distance detected by the loop detector is: $L + L_0$

(L: length of the vehicle, L_0: length of the detector)

- Macroscopic traffic flow variables

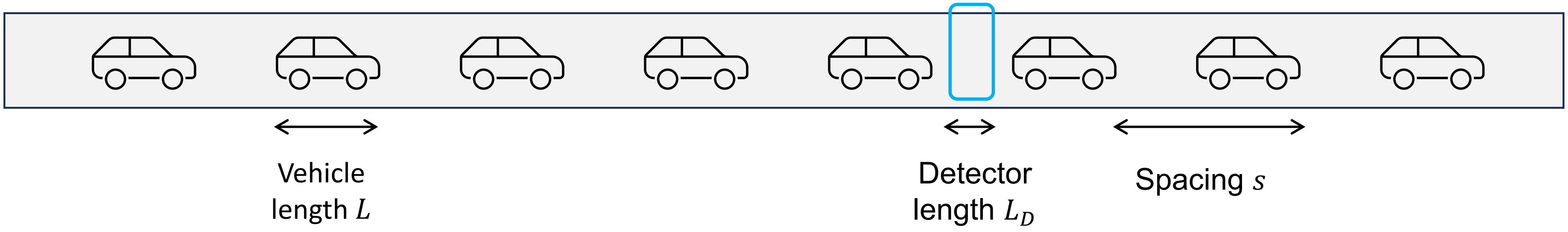
$$\bar{q} = \frac{Count}{\Delta T} \times 3600 \quad [veh/hour, vph]$$

$$\bar{k} = \frac{5280 \times Occupancy}{100 \times (L + L_D)} \quad [veh/mile]$$

$$\bar{v} = \frac{\bar{q}}{\bar{k}} = \frac{68.18 \times (L + L_D)}{\Delta T} \times \frac{Count}{Occupancy} \quad [mile/hour, mph]$$

- L, L_D: in units of feet
- ΔT: time interval (resolution) of the loop detector, in unit of seconds
- Occupancy: %

Inferring densities from occupancy



$$Occupancy(\%) = \frac{L + L_D}{s} \times 100\%$$

Recap: $\bar{s} \cdot \bar{k} = 1$

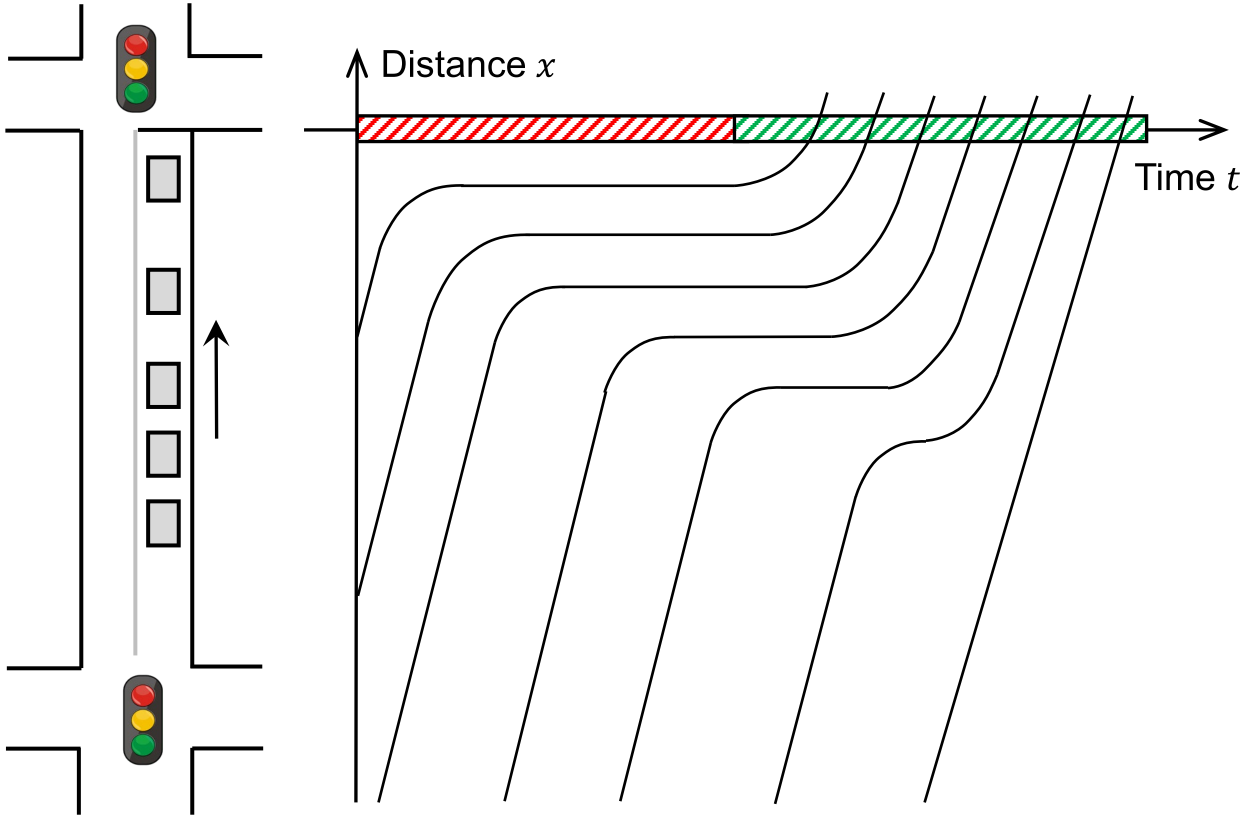
$$\bar{k} = \frac{1}{\bar{s}} = \frac{Occupancy}{L + L_D} \Rightarrow \bar{k} = \frac{5280 \times Occupancy}{100 \times (L + L_D)} \quad [veh/mile]$$

(international unit, IU)

- $Occupancy$: % (5 as 5%)
- L, L_D: feet

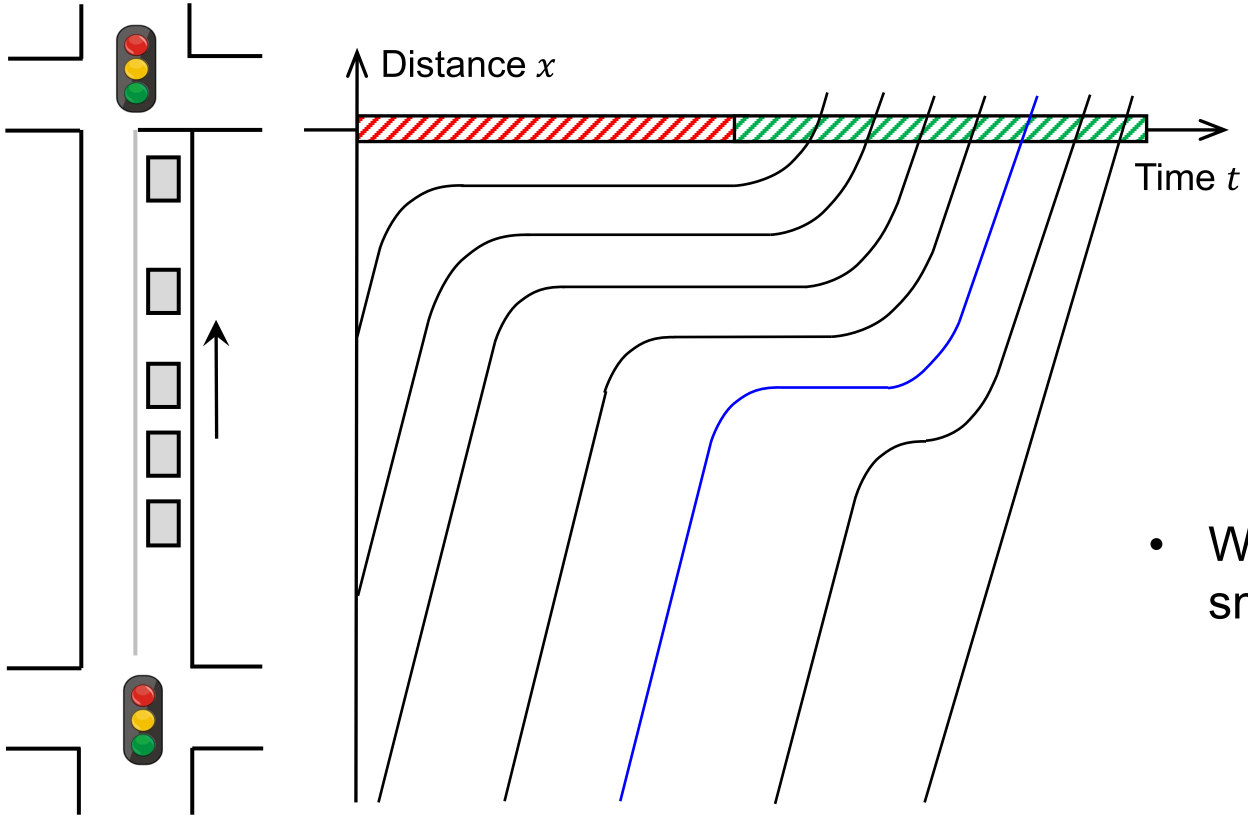
Vehicle trajectory data

- If we can collect the vehicle trajectories of all vehicles, we can get the complete traffic state (location of each individual vehicle, headway, and spacing)



Vehicle trajectory data

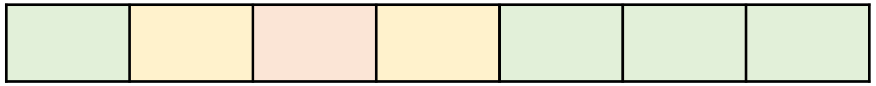
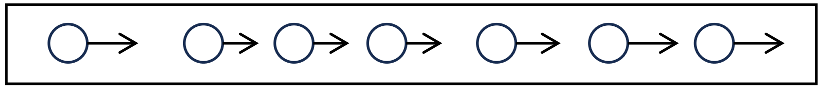
- If we can collect the vehicle trajectories of all vehicles, we can get the complete traffic state (location of each individual vehicle, headway, and spacing)



- What if we can only observe a small proportion of vehicles?

Eulerian vs. Lagrangian

- These two terminologies are borrowed from fluid dynamics

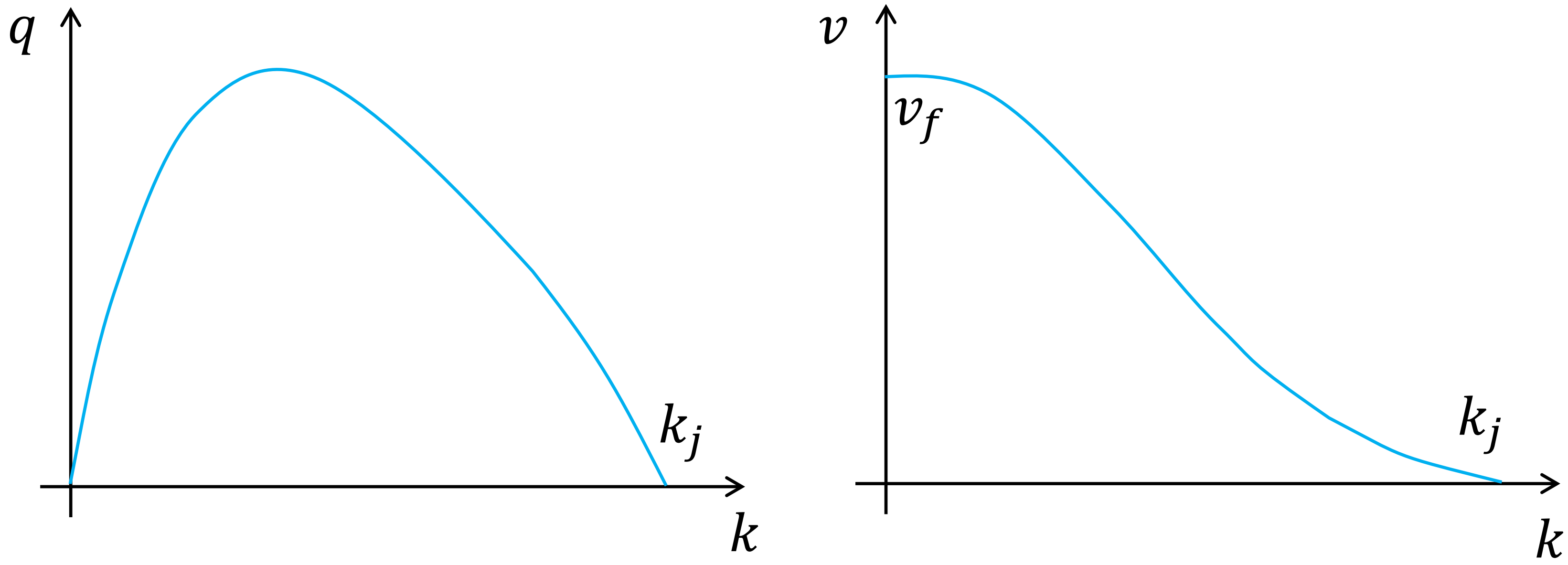
	Eulerian	**Lagrangian**
Explanation	Track the traffic state at certain location and time (x, t) 	Track the movement of individual vehicle 
Traffic flow variables	Average traffic flow, density and speed $\bar{q}(x,t), \bar{k}(x,t), \bar{v}(x,t)$	Location of individual vehicle: $x_j(t)$ Headway and spacing: $h_j(t), s_j(t)$
Scale	Macroscopic	Microscopic (can also be macro)
Measurement	Fixed-location detectors	Vehicle trajectories

Outline

- ❑ Traffic flow variables
- ❑ Measurement of traffic states
- ❑ Fundamental diagram
- ❑ Conservation law

Fundamental Diagram (FD)

- Fundamental diagram: empirical relation between flow, density and speed
- Two basic relations: speed-density relation $V_e\left(\bar{k}(x,t)\right)$ and flow-density relation $Q_e\left(\bar{k}(x,t)\right)$ (subscript e means empirical)



- v_f: free-flow speed
- k_j: jam density
- Monotonically decreasing $v - k$ curve

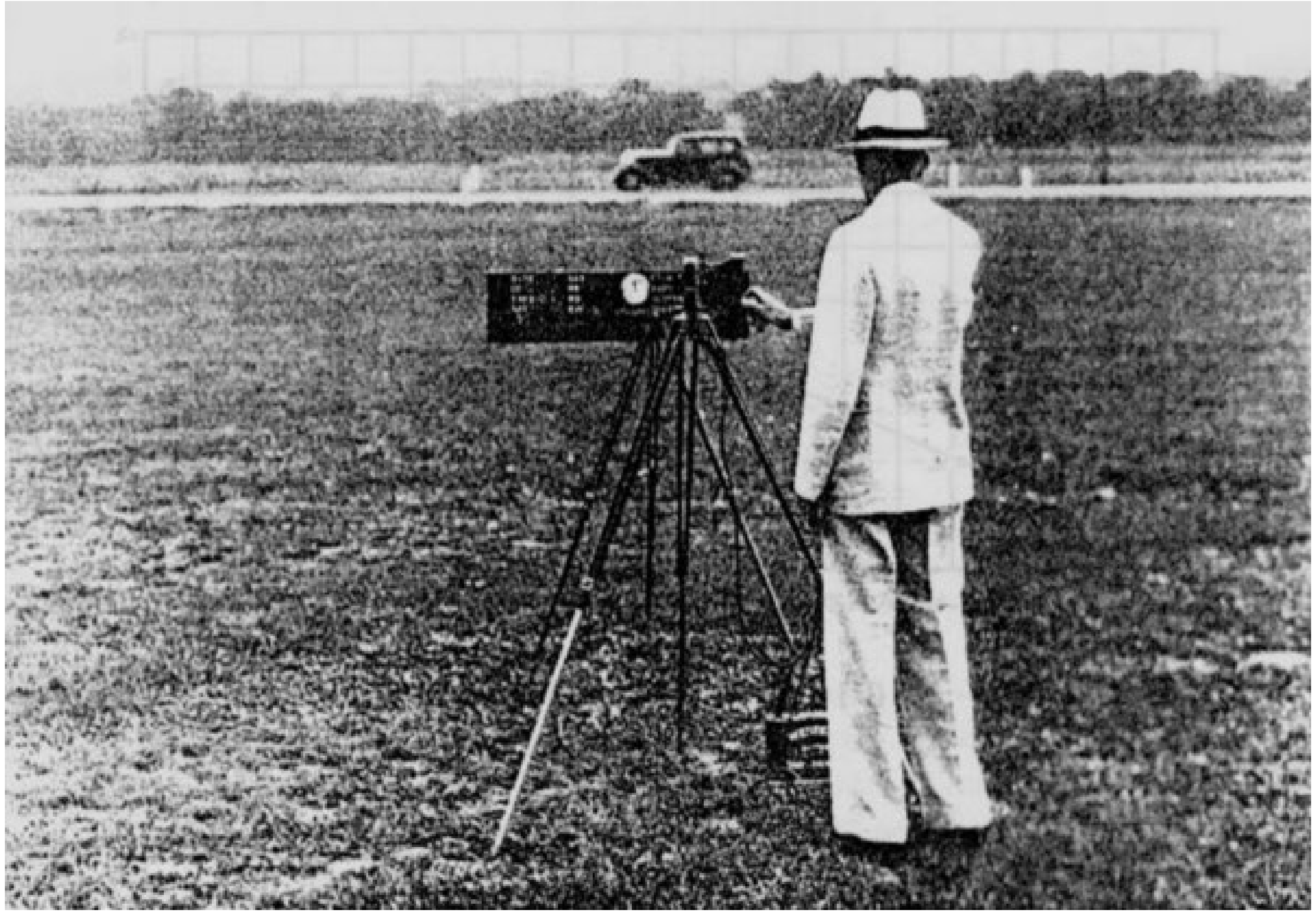
(General trend of fundamental diagram)

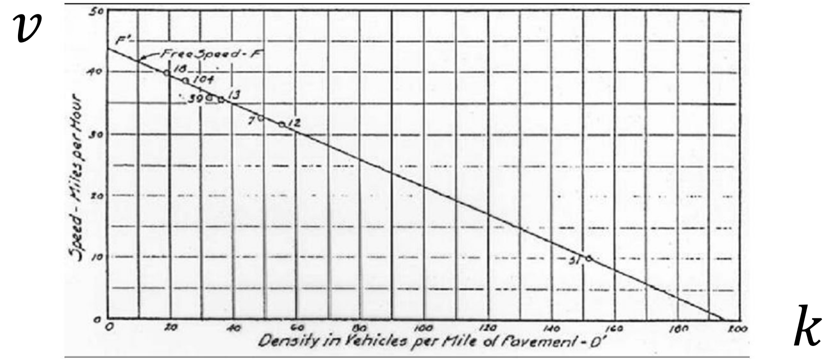
Obtaining fundamental diagram

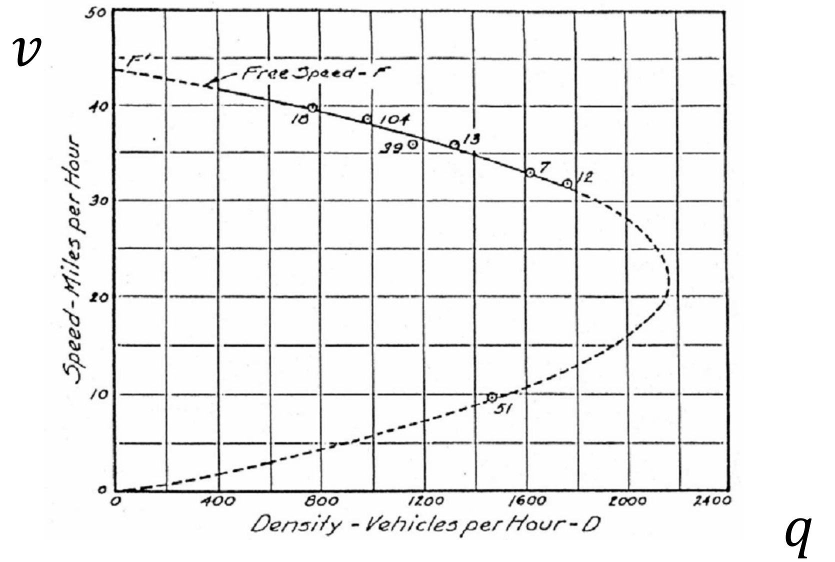
- From direct measurement
  - Video camera, loop detectors, and vehicle trajectories

- Derive from car-following models
  - Newell's simplified car-following model → triangular FD

Empirical relations: Greenshields' Model (1934)







Bruce D. Greenshields (1893-1979)

A brief history of Bruce D. Greenshields

- Bruce D. Greenshields, Ph.D., Civil Engineering, 1934
- Father of traffic flow theory. Pioneering work on traffic flow measurements. Found empirical relationship among traffic flow variables, which is known as “fundamental diagram”

FEBRUARY 9, 2024

U-M CEE Prof. Henry Liu Formally Recognized as the Bruce D. Greenshields Collegiate Professor of Engineering

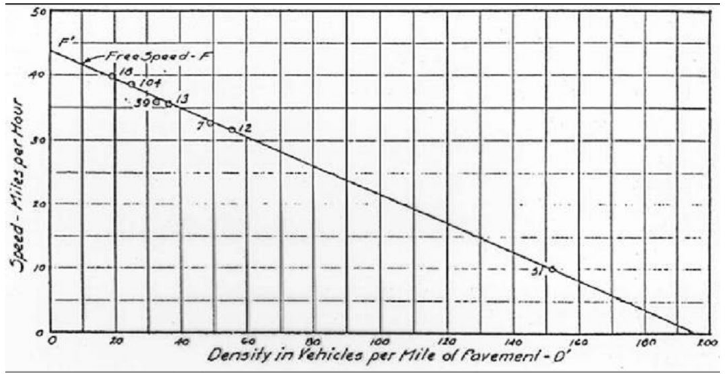
Dr. Liu chose Bruce D. Greenshields when he was recognized as collegiate professor of COE.

Reference:

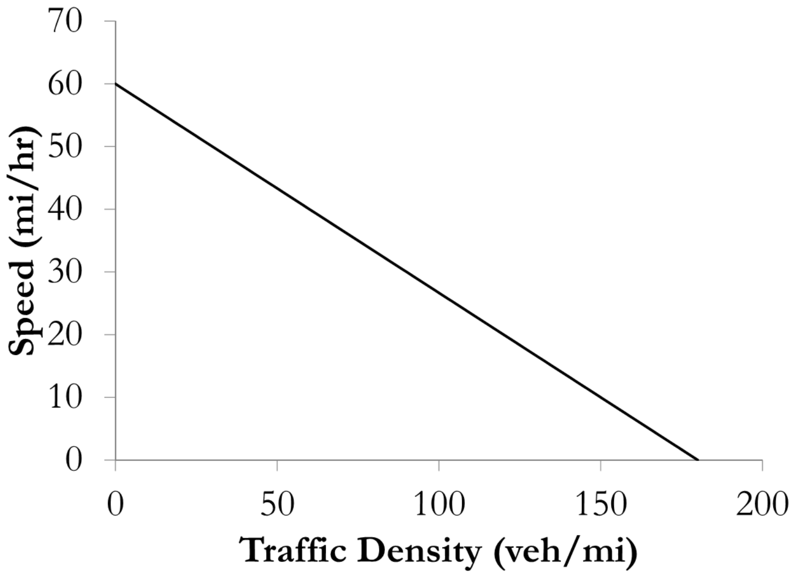
- Kühne, Reinhart. "Foundation of traffic flow theory I: Greenshields legacy highway traffic." *Proceedings Symposium on the Fundamental Diagram: 75 years*. 2008.
- https://cee.engin.umich.edu/2024/02/09/u-m-cee-prof-henry-liu-formally-recognized-as-the-bruce-d-greenshields-collegiate-professor-of-engineering/

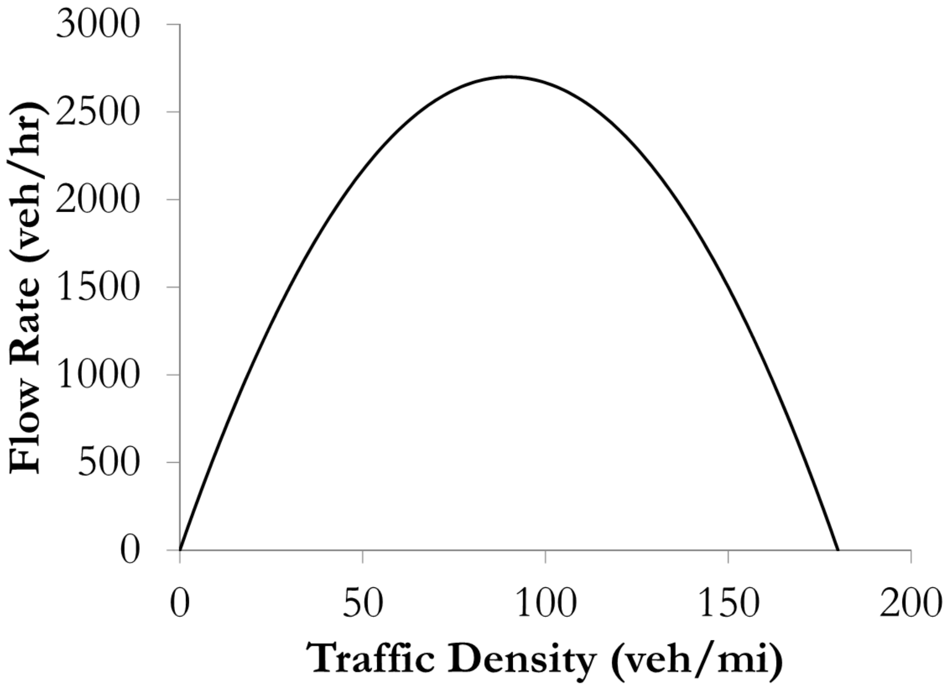
Greenshields' model

- Linear relationship between speed and density



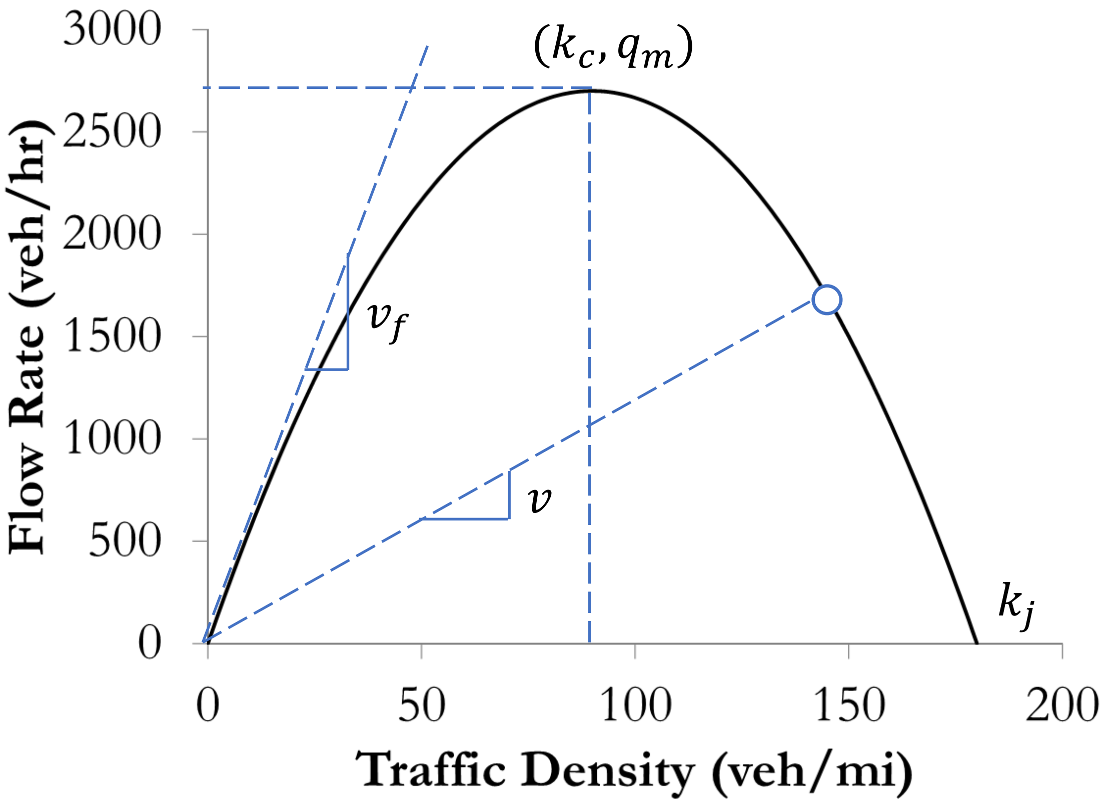
$$V_e(\bar{k}) = v_f\left(1 - \frac{\bar{k}}{k_{jam}}\right) \qquad Q_e(\bar{k}) = v_f\bar{k}\left(1 - \frac{\bar{k}}{k_{jam}}\right)$$





Greenshields' model: $q - k$ relationship

- $q - k$ (flow-density) relationship is the most widely used FD relationships among all three traffic flow variables (q, k, v)



$$Q_e(\bar{k}) = v_f \bar{k}\left(1 - \frac{\bar{k}}{k_{jam}}\right)$$ (a parabola)

$$\frac{\partial q}{\partial k} = 0 \quad \Rightarrow \quad k_c = \frac{k_j}{2}$$

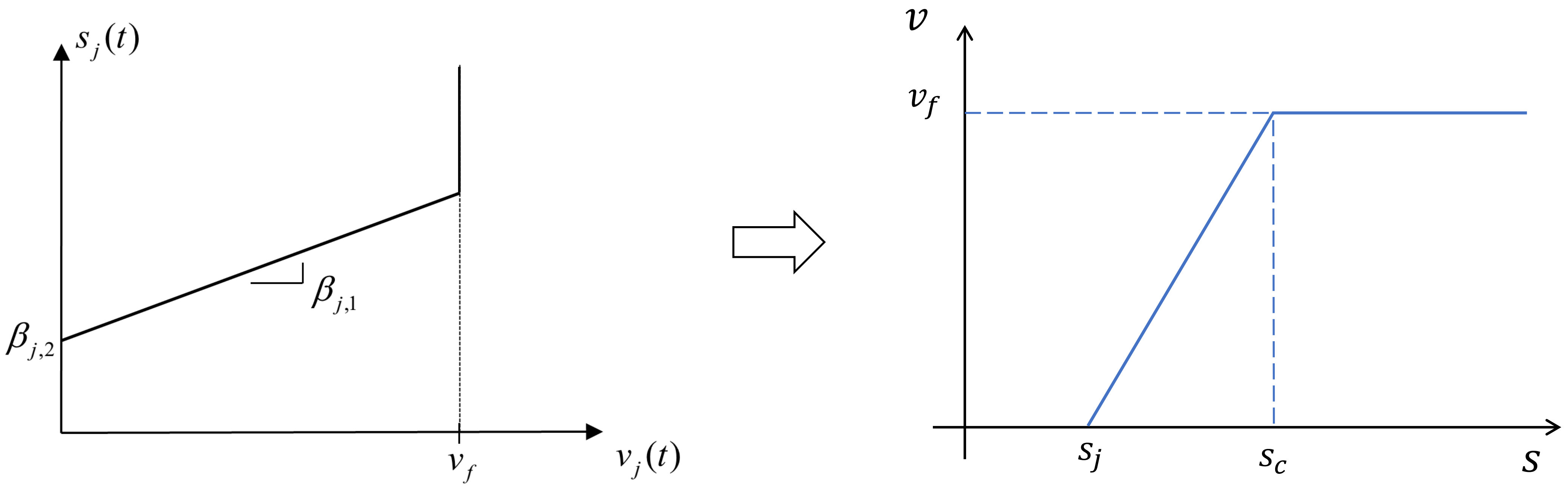
v_f	Free-flow speed
k_c	Critical density
q_m	Maximum flow
k_j	Jam density

Newell's simplified car-following model

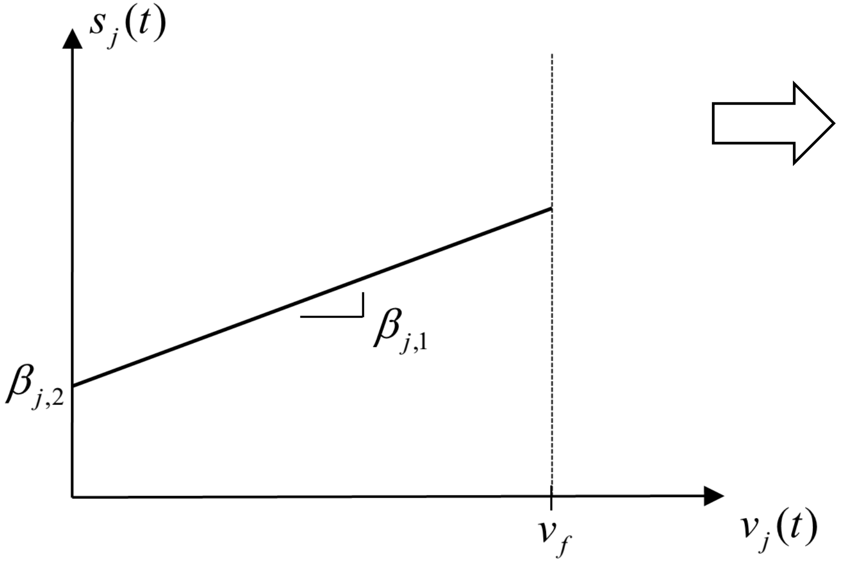
- Newell (2002) assumed a linear relationship between vehicle spacing and speed:

$$s_j(t) = \beta_{j,2} + \beta_{j,1} v_j(t) \qquad 0 \le v_j(t) < v_f$$

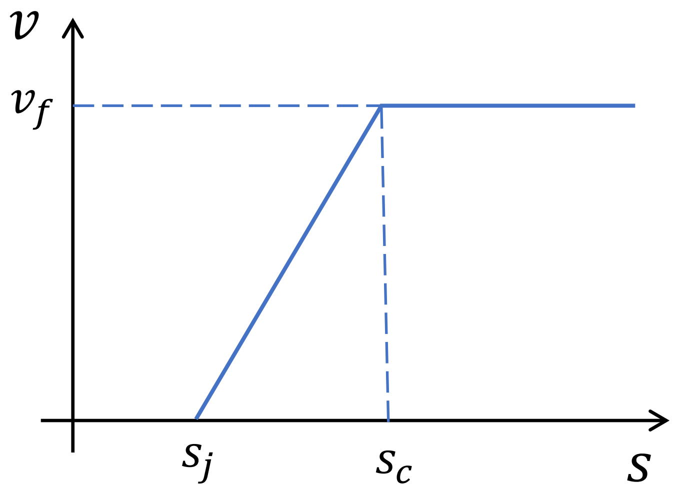
where $\beta_{j,1}$ and $\beta_{j,2}$ are driver-specific parameters.



Deriving FD with Newell's car-following model



$$s_j(t) = \beta_2 + \beta_1 v_j(t)$$
$$0 \le v_j(t) < v_f$$



$$v(s) = \begin{cases} 0, & s \le s_j \\ \dfrac{(s-\beta_2)}{\beta_1}, & s_j < s < s_c \\ v_f, & s \ge s_c \end{cases}$$

$$s_j = \beta_2, \qquad s_c = v_f \beta_1 + \beta_2$$

$$k = \frac{1}{s}, \qquad q = kv$$

$$v(k) = \begin{cases} 0, & k \ge k_j \\ \dfrac{1}{\beta_1 k} - \dfrac{\beta_2}{\beta_1}, & k_c < k < k_j \\ v_f, & k \le k_c \end{cases} \qquad q(k) = \begin{cases} 0, & k \ge k_j \\ \dfrac{1}{\beta_1} - \dfrac{\beta_2}{\beta_1} k, & k_c < k < k_j \\ v_f k, & k \le k_c \end{cases}$$

$$k_j = \frac{1}{s_j} = \frac{1}{\beta_2}. \qquad k_c = \frac{1}{s_c} = \frac{1}{v_f \beta_1 + \beta_2}$$

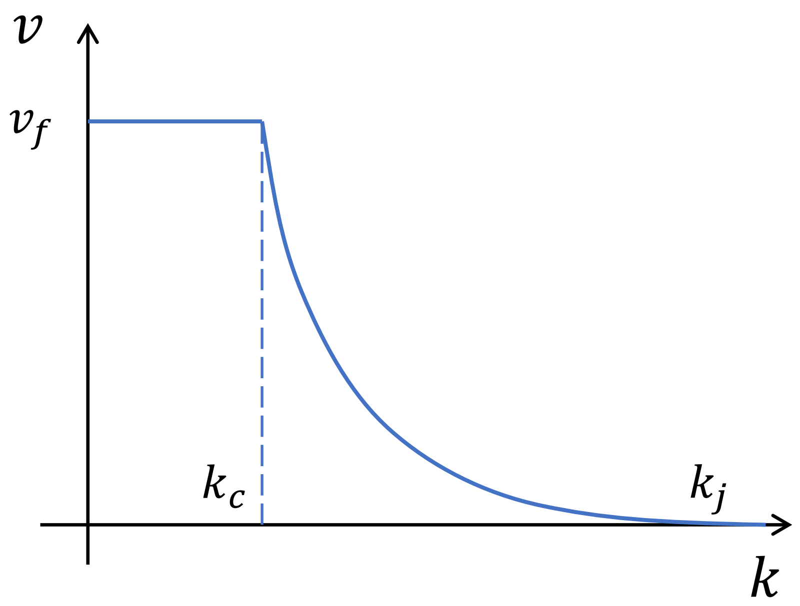
Triangular fundamental diagram

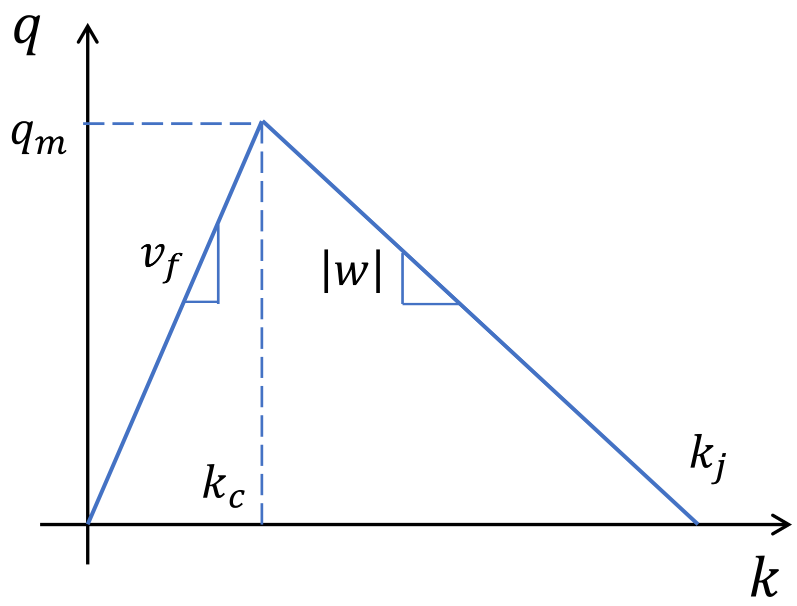
$$v(k) = \begin{cases} 0, & k \geq k_j \\ \dfrac{1}{\beta_1 k} - \dfrac{\beta_2}{\beta_1}, & k_c < k < k_j \\ v_f, & k \leq k_c \end{cases}$$

$$q(k) = \begin{cases} 0, & k \geq k_j \\ \dfrac{1}{\beta_1} - \dfrac{\beta_2}{\beta_1} k, & k_c < k < k_j \\ v_f k, & k \leq k_c \end{cases}$$

$$k_j = \frac{1}{s_j} = \frac{1}{\beta_2}.$$

$$k_c = \frac{1}{s_c} = \frac{1}{v_f \beta_1 + \beta_2}$$





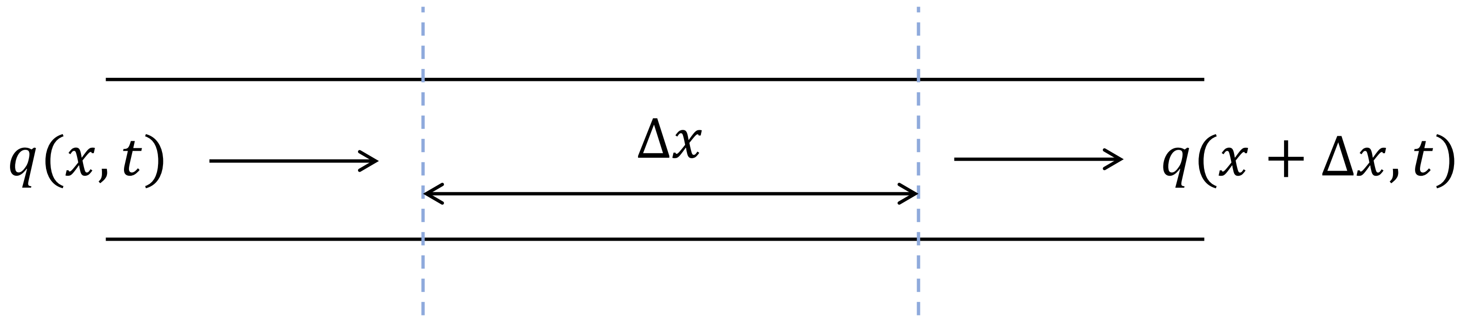
v_f	Free-flow speed
k_c	Critical density
q_m	Maximum flow
k_j	Jam density
w	Shockwave speed

Outline

- ❑ Traffic flow variables
- ❑ Measurement of traffic states
- ❑ Fundamental diagram
- ❑ Conservation law

Conservation law

- For an arbitrary road segment $[x, x + \Delta x]$



$$k(x, t + \Delta t)\Delta x = k(x, t)\Delta x + q(x, t)\Delta t - q(x + \Delta x, t)\Delta t$$

$$\Rightarrow \quad \frac{k(x, t + \Delta t) - k(x, t)}{\Delta t} + \frac{q(x + \Delta x, t) - q(x, t)}{\Delta x} = 0$$

$$\Rightarrow \quad \frac{\partial k(x, t)}{\partial t} + \frac{\partial q(x, t)}{\partial x} = 0$$

Traffic flow relationship

- Traffic flow physics

$$q(x,t) = k(x,t) \cdot v(x,t)$$

- Conservation law

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0$$

- Empirical observation (fundamental diagram)

$$q(x,t) = Q_e\big(k(x,t)\big)$$

Reading

- TFT_Document.pdf, "First-Order Macroscopic Traffic Flow Modeling", Section 1-4
- Kühne, Reinhart. "Foundation of traffic flow theory I: Greenshields legacy highway traffic." *Proceedings Symposium on the Fundamental Diagram: 75 years*. 2008.
- Newell, Gordon Frank. "A simplified car-following theory: a lower order model." *Transportation Research Part B: Methodological* 36.3 (2002): 195-205.

Homework assignment

- Problem 1 and Problem 2 of Homework 1
- No due time yet